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Comparison of Thermal Modal Analysis and Proper Orthogonal Decomposition methods for thermal error estimation

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Abstract

Due to the increasing demand on more accurate products without losing high rates of productivity, the control of thermal induced errors in machine tools is still a relevant issue. To compensate the errors from thermal deformation, the temperature field in the machine needs to be known. In order to reproduce thermal fields, accurate models are needed, but not only that, apart from accurate, models are required to be of low computational effort. To overcome the issue of high time-consuming models, model order reduction techniques have been applied to thermos-elastic models. Moreover, it is necessary to obtain the maximum information as possible from experimental tests. Reduced order models are useful for this too. For all the aforementioned, this work proposes a methodology to evaluate the capabilities and limitations of two model order reduction techniques (thermal modal analysis and proper orthogonal decomposition) and their application to optimal sensor positioning and their robustness considering different load cases and conditions.

Thermal modal analysis, Proper Orthogonal Decomposition

1. Introduction

Controlling errors due to thermal deformations is one of the biggest issues in machine tools due to the demand of more accurate products (straighter tolerances and better quality) and keeping high productivity [1].

In order to predict the thermal induced errors, it arises the need of reproducing the thermal field. To do that, from experimental test it is necessary to acquire the maximum amount of information as possible with the minimum number of inputs. To do that, reducer order models are really useful.

Some works have already been published dealing with model reduction techniques and methods for optimal temperature sensor placement. Koevets et al. [2] proposed a methodology based on comparing the optimal sensor positioning using nodal and modal compensation methods. Benner et al. [3] compared several model order reduction methods to find an optimal sensor positioning. In these works, thermal modal analysis and proper orthogonal decomposition methods are used.

There are several mathematical methods to find the optimum sensor positions for thermo-elastic models. Research on thermal modal analysis has been performed on modelling of machine tools [4], on thermal error compensation on machine tools [5] and other applications [6], since its content on physical meaning helps to characterise well a thermo-elastic system. Proper orthogonal decomposition is a pure mathematical method [7,8] that works well when the conditions are well-known but does not extract all the physical content of the thermo-elastic system.

In this work, a theoretical example of a 2D plate is proposed to compare the reduction methods of thermal modal analysis and proper orthogonal decomposition. The aim of the proposed study is to compare both methods and understand their capabilities and limitations by means of a sensitivity analysis. The objective is to understand in which applications or cases is more suitable each method. This builds a step towards a further analysis on optimal sensor positioning by means of these methods.

2. Thermal modal analysis and Proper Orthogonal Decomposition for thermo-elastic models

The case of study in this work is a square 2D plate fixed in one side, represented in Figure 1. The dimensions of the 2D plate are 250mmx250mm, and two load positions are proposed for evaluation.

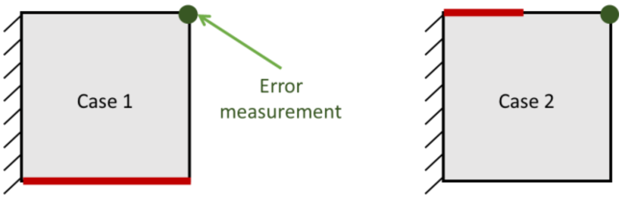


Figure 1. Case study: square 2D plate fixed in one side

The equation for calculating the thermal field may be written as

$$\boldsymbol{C^t}\dot{\boldsymbol{\theta}}(t) + \boldsymbol{K^t}\boldsymbol{\theta}(t) = \boldsymbol{q}(t) \qquad (1)$$

Where $\boldsymbol{C^t}$ is the specific heat or thermal inertia matrix, $\boldsymbol{K^t}$ is the conductivity matrix, $\boldsymbol{q}(t)$ is the thermal load vector and $\boldsymbol{\theta}(t)$ is the temperatures vector. The equation that couples the thermal with the elastic behaviour, when there are not included mechanical loads may be represented as

$$\boldsymbol{K^u}\boldsymbol{u}(t) + \boldsymbol{K^{ut}}\left(\boldsymbol{\theta}(t) - \boldsymbol{\theta_{ref}}\right) = \boldsymbol{0} \qquad (2)$$

where $\boldsymbol{K^u}$ is the stiffness matrix, $\boldsymbol{K^{ut}}$ is the thermoelastic stiffness matrix, $\boldsymbol{u}(t)$ is the displacements vector, $\boldsymbol{\theta}(t)$ is the temperatures vector and $\boldsymbol{\theta_{ref}}$ is the initial temperature vector.

Then the equation for a coupled thermoelastic analysis which results as combination of the Eqs. *(1)* and *(2)*, may be written in matrix form as:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{C^t} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}(t) \\ \dot{\boldsymbol{\theta}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{K^u} & \boldsymbol{K^{ut}} \\ \mathbf{0} & \boldsymbol{K^t} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{\theta}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{K^{ut}\theta_{ref}} \\ \boldsymbol{q} \end{bmatrix} \boldsymbol{z}(t) \qquad (3)$$

where $\boldsymbol{C^t}$ is the specific heat or thermal inertia matrix, $\boldsymbol{K^t}$ is the conductivity matrix, $\boldsymbol{K^u}$ is the stiffness matrix, $\boldsymbol{K^{ut}}$ is the thermoelastic stiffness matrix, $\boldsymbol{q}$ is the thermal load vector, $\boldsymbol{u}(t)$ is the displacements vector and $\boldsymbol{\theta}(t)$ is the temperatures vector. Since the quantity of interest are the thermal deformation, the solution results as

$$\boldsymbol{y(t)} = \begin{bmatrix} \boldsymbol{O^u} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{O^t} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}(\boldsymbol{t}) \\ \dot{\boldsymbol{\theta}}(\boldsymbol{t}) \end{bmatrix} \qquad (4)$$

where $\boldsymbol{O^u}$ and $\boldsymbol{O^t}$ are the matrices that link the temperature and the displacement with the state-space output vector.

Eqs. *(3)* and (*4)* may be re-written in state-space form as follows

$$\begin{cases} \dot{\boldsymbol{x}}(\boldsymbol{t}) = -\boldsymbol{C^{-1}Kx(t)} + \boldsymbol{C^{-1}fz(t)} \\ \boldsymbol{y(t)} = \boldsymbol{Ox(t)} \end{cases} \qquad (5)$$

where $\boldsymbol{x}(t)$, $\boldsymbol{z}(t)$ and $\boldsymbol{y}(t)$ are the state vector, the input and the output respectively.

2.1. Thermal modal analysis

The discrete heat equation from Eq. *(3)* may be derived and may be written as

$$\boldsymbol{C^t}\,\frac{d\,\boldsymbol{\theta}(t)}{dt} + \boldsymbol{K^t}\,\boldsymbol{\theta}(t) \;=\; \boldsymbol{q}(t). \qquad (6)$$

The system in *Eq. (6)* is a system of *n* first order linear differential equation with constant coefficients. So, the general solution of the system is the sum of the general solution of the homogeneous system plus a particular solution of the whole system.

The homogeneous system is

$$\boldsymbol{C^t}\,\frac{d\,\boldsymbol{\theta}(t)}{dt} + \boldsymbol{K^t}\,\boldsymbol{\theta}(t) \;=\; \mathbf{0} \qquad (7)$$

whose solutions are

$$\boldsymbol{\theta}(t) \;=\; \boldsymbol{\Phi}\, e^{-\lambda\, t}, \qquad (8)$$

replacing them in the homogeneous system, it results

$$(\boldsymbol{K^t} - \lambda \boldsymbol{C^t})\boldsymbol{\Phi} \;=\; \mathbf{0}. \qquad (9)$$

This system is a generalised problem of eigenvalues and eigenvectors. The characteristic equation of this system is $|\boldsymbol{K^t} - \lambda\boldsymbol{C^t}| = \mathbf{0}$. The solution of the characteristic equation gives *n* eigenvalues $\lambda_1, \lambda_2, .., \lambda_n$. The eigenvalues are the inverse of the time constantans of each mode $\tau_1, \tau_2, \ldots, \tau_n$. Replacing the eigenvalues in the system and solving it is obtained a series of eigenvectors $\boldsymbol{\Phi_i}$ associated to them. So, the general equation of the homogeneous system results

$$\boldsymbol{\theta}(t) \;=\; \sum_{i=1}^{n} \boldsymbol{D_i}\,\boldsymbol{\Phi_i}\, e^{-\lambda_i\, t} \qquad (10)$$

The particular solution of the whole system depends on the shape of the thermal excitations $\boldsymbol{q}(t)$, and it results

$$\boldsymbol{\theta}(t) \;=\; \boldsymbol{K^{t^{-1}}}\,\boldsymbol{q}(t) \qquad (11)$$

thus, the general solution of the system is

$$\boldsymbol{\theta}(t) \;=\; \boldsymbol{K^{t^{-1}}}\,\boldsymbol{q}(t) + \sum_{i=1}^{n} \boldsymbol{D_i}\,\boldsymbol{\Phi_i}\, e^{-\lambda_i\, t} \qquad (12)$$

The study of a discrete thermal system can be reduced to the resolution of a classic problem of eigenvalues and eigenvectors as

$$\boldsymbol{K^t}\,\boldsymbol{V_i} \;= \lambda_i\; \boldsymbol{C^t}\,\boldsymbol{\Phi_i} \qquad (13)$$

The eigenvectors $\boldsymbol{\Phi_i}$ form a system of *n* lineally independent vectors that constitute a base. The temperature vector $\boldsymbol{\theta}(t)$ can be expressed related to this base with some new coordinates $\boldsymbol{\xi}(t)$, that is

$$\boldsymbol{\theta}(t) \;=\; \textstyle\sum_{i=1}^{n} \boldsymbol{\Phi_i}\xi_i(t) \;=\; \boldsymbol{\Phi\xi}(t). \qquad (14)$$

Replacing Eq. *(14)* int the original system of Eq. *(6)* and multiplying by $\boldsymbol{\Phi}^{\mathrm{T}}$

$$\boldsymbol{\Phi^T C^t \Phi}\dot{\boldsymbol{\xi}}(t) \;+ \boldsymbol{\Phi^T K^t}\,\boldsymbol{\Phi\xi}(t) \;= \boldsymbol{\Phi^T}\,\boldsymbol{q}(t), \qquad (15)$$

then, if $\boldsymbol{\Phi^T}\,\boldsymbol{q}(t) \;=\; \boldsymbol{\psi}(t)$, Eq. *(15)* may be re-written as

$$\dot{\boldsymbol{\xi}}(t) \;+ \boldsymbol{\lambda\xi}(t) \;= \boldsymbol{\psi}(t), \qquad (16)$$

since $\boldsymbol{\Phi^T C^t \Phi} = \mathbf{0}$ and $\boldsymbol{\Phi^T K^t}\,\boldsymbol{\Phi} = \boldsymbol{\lambda}$ due to the orthogonality of the eigenvectors respect to the matrices $\boldsymbol{K^t}$ and $\boldsymbol{C^t}$. As $\boldsymbol{\lambda}$ is a diagonal matrix, immediately it is observed that in Eq. *(16)* there is a system of *n* uncoupled equations with the form

$$\dot{\boldsymbol{\xi}}(t) \;+ \lambda_i\; \boldsymbol{\xi}(t) \;= \boldsymbol{\psi_i}(t). \qquad (17)$$

Thus, with $\boldsymbol{\xi}(t)$, denominated natural coordinates, the system of *n* differential equations with *n* variables becomes *n* equations of a single variable.

The eigenvectors $\boldsymbol{\Phi_i}$ are the so-called natural thermal modes, and the methodology employed for uncoupling equations of thermal balance is called Thermal Modal Analysis (TMA).

2.2. Proper Orthogonal Decomposition

Proper Orthogonal Decomposition (POD) method is based on the SVD computation of a matrix that contains various vector results (from simulations or experimental measurements). These vectors are denominated snapshots. Considering the linear dynamical system in Eq. *(5)*, for a fixed input $\boldsymbol{z}(t)$, the state trajectory $\boldsymbol{x}(t)$ at certain instants t_k is measured as

$$\boldsymbol{\chi} = \left[\boldsymbol{x}(t_1)\;\; \boldsymbol{x}(t_2) \ldots \boldsymbol{x}\left(t_{n_{snap}}\right)\right] \in \mathbb{R}^{n \times nsnap} \qquad (18)$$

being $\boldsymbol{\chi}$ the matrix of snapshots and n_{snap} the number of snapshots, in general $n_{snap} \geq n$. The singular value decomposition of $\boldsymbol{\chi}$ is performed and if the singular values of this matrix fall off rapidly, a low-order k approximation of this system may be computed

$$\boldsymbol{\chi} = \boldsymbol{USV^*} \approx \boldsymbol{U_k S_k V_k}, \quad k \ll n. \qquad (19)$$

The eigenvalues of $\boldsymbol{\chi}$ typically decay exponentially for snapshots drawn from the heat equation. The truncation threshold to obtain k is chosen such that the ratio of the energies contained in the bases of the reduced and full models is near 1, that is,

$$\frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{n_{snapshots}} \lambda_i} = \frac{\sum_{i=1}^{k} \lambda_i}{trace(\boldsymbol{S})} \approx 1, \qquad (20)$$

being λ_i the *i-th* component of the diagonal of the matrix $\boldsymbol{S}$.

Then the model reduction is applied to the state space $\boldsymbol{x}(t)$ as follows

$$\boldsymbol{x}(t) \approx \boldsymbol{S_k}\hat{\boldsymbol{x}}(t) \qquad (21)$$

where $\boldsymbol{S_k} \in \mathbb{R}^{n \times k}$ and $\hat{\boldsymbol{x}}(t) \in \mathbb{R}^k$.

The main problem of this method is that the resulting simplification depends on the initial excitation imposed to the system, the obtained singular values are not system invariants. The main advantage arises from the applicability to high-complexity linear and nonlinear systems.

3. Methodology

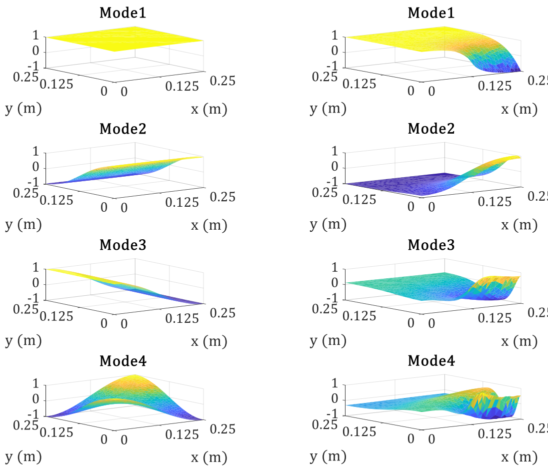
In this section, a methodology to find the optimum sensor placement for the mathematical methods of thermal modal analysis and proper orthogonal decomposition is proposed. The presented methodology considers physical meaning and practical/operational viability.

3.1. Modes comparison

In order to stablish a reasonable criterion for both methods, the thermal modes of thermal modal analysis are compared with the modes of the base $\boldsymbol{S}_{\boldsymbol{k}}$ from proper orthogonal decomposition method.

The main difference between the modes from thermal modal analysis $\boldsymbol{\Phi}_{\boldsymbol{i}}$ and the ones from POD $\boldsymbol{S}_{\boldsymbol{k}_{\boldsymbol{i}}}$ is the physical meaning or content. In the case of thermal modal analysis, the thermal modes contain information from the physical system and are independent from the thermal load imposed to the system. In the case of POD, the thermal modes depend on the matrix of snapshots $\boldsymbol{\chi}$.

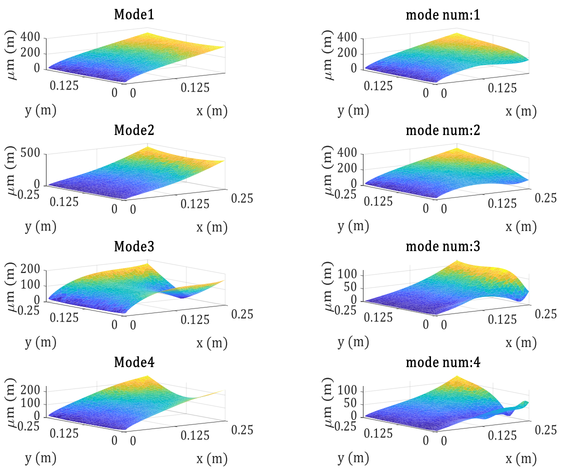
In Figure 2 are shown the first four modes of thermal modal analysis (a) and from POD (b). As it may be observed, the modes shape is completely different. In Figure 2 (b) it is seen that the modes shape of POD depends on the input load.



(a) (b)

Figure 2. First four thermal modes from thermal modal analysis (a) and from POD (b).

Figure 3 shows the thermal growth that corresponds to the modes in both cases thermal modal analysis (a) and POD (b). As in the case of the thermal modes the growth behaviour for this case of study for POD Figure 3 (b) depends on the input load location, not like in the thermal modal analysis Figure 3 (b) case that depends on the system.



(a) (b)

Figure 3. First four growth modes from thermal modal analysis (a) and from POD (b).

The shape of the thermal modes is going to be used to select the sensor position, in order to optimise the quantity of information acquired.

3.2. Base reduction and number of sensors

The number of relevant modes or order to reduce the base depend on the method. In the case of proper orthogonal decomposition, the reduction is related with the truncation condition in Eq. *(20)*. In the case of thermal modal analysis, the modes are order considering the value of the time constants of each mode τ_i.

3.3. Sensor position selection

Since in real physical systems there exists limitations, (due to shapes, functionalities, …) the first consideration to be made it is to impose limits to the possible sensor locations. To do that, the first step is to divide the body into cells where there could exist the possibility to place the sensors. The sensor location is going to be selected where the maximums or minimums of the most significant modes are located. This would give some physical meaning and content to the problem. Figure 4 shows an example of how the sensor placement pre-selection would be.

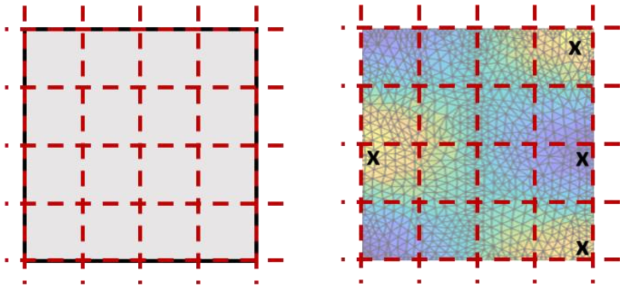


Figure 4. Available positions for sensor location and sensor position location based on absolute maximums of the modes.

For each mode a certain number of absolute maximums is needed, that would mean number of sensors by mode, and the quantity of modes to be used has to be defined or selected. To do so, the information that contains each node and its weight has to be evaluated.

4. Results and analysis

In order to evaluate the capabilities of thermal modal analysis and proper orthogonal decomposition methods, two sensitivity analysis are proposed. The first one is focused on the modal base reduction and the second one with the optimal sensor positioning.

To perform the sensitivity analysis, the load position of case 1 from Figure 1 with a step input is selected to create the base of both methods. From the base obtained with this analysis, several load cases (unit step, sinusoidal) will be evaluated performing variations on amplitudes and frequencies. Moreover, the accuracy of the reduced models when the load position is modified will be evaluated.

In the case of optimal sensor positioning, a comparative analysis of optimal sensor placement will be performed. The objective of this analysis is to understand which method (POD or TMA) is more adequate depending on the case under study.

As a result of the methodology proposed in Section 3, the capabilities and limitations of thermal modal analysis and proper orthogonal decomposition as model order reduction techniques are going to be stated. Furthermore, an evaluation of the capacities and versatility of both methods regarding optimal sensor positioning will be performed.

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