Uncertainty associated with coordinate measurement in comparator mode

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Abstract

Coordinate measuring systems (CMSs) such as coordinate measuring machines (CMMs) are used extensively in industry for form and tolerance assessment. However, the evaluation of the uncertainty associated with CMM measurement, for example, is not straightforward as CMM measurement results are subject to a large range of influence factors that are difficult to quantify. Coordinate measuring systems can also be used in comparator mode in order to compare the geometry of a workpiece or test artefact with that of a master artefact nominally of the same design. The advantage of operating in comparator mode is that many of the systematic effects associated with the measurement system apply both to the measurement of the test artefact and the master artefact and, in evaluating the differences in the two artefacts, a substantial proportion of the systematic effects associated with the two sets of measurements cancel out. In this way, the calibration information associated with the master artefact can be transferred to the test artefact with only a modest increase in uncertainty due to the comparison measurements. The purpose of this paper is to describe a methodology for uncertainty evaluation associated with CMS measurement, discuss how this methodology can be applied to CMSs operating in comparator mode and to the collaborative calibration of test artefacts using a calibrated master artefact. This methodology is illustrated on calibrating a cylindrical feature.

1 Introduction

The primary task of form and tolerance assessment in precision engineering is to estimate how close a manufactured workpiece is to its ideal geometry, as specified by a technical drawing or CAD specification. Traditionally, such assessment was made using hard gauges, but in recent decades coordinate measuring systems such as coordinate measuring machines (CMMs) are used extensively in industry for this task. However, the evaluation of the uncertainty associated with CMM measurement, for example, is not straightforward. While the uncertainty methodologies presented in the Guide to the Expression of Uncertainty in Measurement (GUM, [1, 2] can be applied to CMM measurement [9], such an endeavour is made complicated by the large range of multivariate influence factors, including kinematic errors, probing effects and environmental effects, that are difficult to quantify and considerable effort has to be made in order to provide valid uncertainty estimates associated with a particular measuring task.

Coordinate measuring systems can also be used in comparator mode in order to compare the geometry of a workpiece with that of a master artefact nominally of the same shape. For example, many gauge block calibrations involve determining the difference in length between the gauge block under test and that of a calibrated (master) gauge block. The advantage of operating in comparator mode is that many of the systematic effects associated with the measurement system apply both to the measurement of the test artefact and the master artefact and, in evaluating the differences in the two artefacts, a substantial proportion of the systematic effects associated with the two sets of measurements cancel out. In this way, the calibration information associated with the master artefact can be transferred to the test artefact with only a modest increase in uncertainty due to the comparison measurements. In addition, many of the difficulties associated with evaluating the uncertainties associated with coordinate measuring systems operating in absolute mode are largely avoided.

The purpose of this paper is to describe a methodology for uncertainty evaluation associated with comparative measurements. In section 2, we give an overview of the evaluation of the uncertainty associated with comparator measurement involving a single measurand, such as the length of a gauge block. The comparator calibration of a gauge block is covered in the GUM [1] and the basic principles can be applied to CMM measurement [8]. In section 3, we provide a general model for characterising the behaviour of coordinate measuring systems in terms of random and systematic effects, the latter described by empirical functions or spatially correlated effects. In section 4, we show how uncertainties associated with a set of coordinate measurements can be propagated through to the parameters of a best-fit geometric feature. In section 5, we use the uncertainty methodology developed in sections 3 and 4 to assign an uncertainty to comparator measurements and in section 6 we apply this methodology to the collaborative calibration of a test artefact from a calibration of a master artefact. Section 7 gives numerical results for determining a cylindrical feature using collaborative calibration techniques. Our concluding remarks are given in section 8.

1.1 Notation

Given point coordinates $\boldsymbol{x}_i = (x_i, y_i, z_i)^{\mathrm{T}}$, $i \in I = \{1, \ldots, m\}$, $\boldsymbol{x}_I$ is the $3m \times 1$ vector

$$\boldsymbol{x}_I = (x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_m, y_m, z_m)^{\mathrm{T}}.$$

2 Collaborative calibration of a quantity associated with a test artefact

Suppose a measurement system, labelled A, measures a master artefact and a test artefact, nominally the same as the test artefact, giving rise to measurement results x_A^* and x_A modelled according to

$$\begin{aligned} x_A^* &= a^* + e_A^* + \epsilon_A^*, \quad e_A^* \sim \mathrm{N}(0, \tau_A^2), \quad \epsilon^* \in \mathrm{N}(0, \sigma_A^2), \\ x_A &= a + e_A + \epsilon_A, \quad e_A \sim \mathrm{N}(0, \tau_A^2), \quad \epsilon_A \in \mathrm{N}(0, \sigma_A^2). \end{aligned} \tag{1}$$

Here a^* and a are the quantities being measured, e.g., the length of gauge blocks, e_A^* and e_A are systematic effects that are highly correlated with each other with $\mathrm{cov}(e_A^*, e_A) = \rho_A \tau_A^2$, with ρ_A near 1, and ϵ_A^* and ϵ_A are random effects. The variance matrix $V(x_A^*, x_A)$ associated with this pair of measurements is

$$V(x_A^*, x_A) = \left[\begin{array}{cc} \sigma_A^2 + \tau_A^2 & \rho_A \tau_A^2 \\ \rho_A \tau_A^2 & \sigma_A^2 + \tau_A^2 \end{array} \right]. \tag{2}$$

A second system, labelled C, also provides measurements of the master and test artefact according to model similar to (1), providing measurements x_C^* and x_C, with associated variance matrix $V(x_C^*, x_C)$ defined as in (2) but with subscripts C. System C provides the estimate $x_C - x_C^*$ of $a - a^*$ and the variance associated with this estimate is

$$[1 \ -1] V(x_C^*, x_C) [1 \ -1]^{\mathrm{T}} = 2(\sigma_C^2 + (1 - \rho_C)\tau_C^2), \tag{3}$$

so that if ρ_C is close to 1, the uncertainty associated with this estimate is close to $\sqrt{2}\sigma_C$, i.e., is determined by the repeatability of the system.

A collaborative estimate of a can be determined from the measurement of the master artefact by system A and the measurements of the master and test artefact by system C so that a is estimated by $a_{AC} = x_A^* + (x_C - x_C^*)$, with associated uncertainty $u(a_{AC})$ given by

$$u^2(a_{AC}) = \sigma_A^2 + \tau_A^2 + 2(\sigma_C^2 + (1 - \rho_C)\tau_C^2).$$

If ρ_C is close to 1, then $u(a_{AC})$ does not depend strongly on τ_C, so that the systematic effects associated with system C can be large, relative to the repeatability of system C, but contribute little to the uncertainty $u(a_{AC})$.

The measurement x_A of the test artefact by system A can be used to validate the collaborative estimate by evaluating $\Delta a = (x_A - x_A^*) - (x_C - x_C^*)$, with associated uncertainty $u(\Delta a)$ given by

$$u^2(\Delta a) = 2(\sigma_A^2 + (1 - \rho_A)\tau_A^2) + 2(\sigma_C^2 + (1 - \rho_C)\tau_C^2).$$

If ρ_A and ρ_C are close to 1, the systematic effects associated with both systems are largely cancelled out so that $u(\Delta a) \approx (2(\sigma_A^2 + \sigma_C^2))^{1/2}$ and depends mainly on the repeatability of the two systems. This cancellation allows an accurate assessment of the validity of the collaborative estimate a_{AC} of a: the uncertainty $u(\Delta a)$ applies when assessing the performance of a comparator against an absolute system when both systems are used in comparator mode.

3 A spatial correlation model for coordinate measurement

We assume that data points $\boldsymbol{x}_I$ gathered by a CMS are generated according to a model of the form

$$\boldsymbol{x}_i = \boldsymbol{s}_i + \boldsymbol{e}(\boldsymbol{s}_i, \boldsymbol{b}) + \boldsymbol{f}_i + \boldsymbol{\epsilon}_i, \quad i \in I = \{1, \ldots, m\}, \tag{4}$$

where $\boldsymbol{s}_i$ is the true data point related to the surface of an artefact, $\boldsymbol{e}(\boldsymbol{s}_i, \boldsymbol{b})$ represents a model of the systematic effects associated with the CMS, depending on a $p \times 1$ vector of parameters $\boldsymbol{b}$, $\boldsymbol{f}_i$ is a spatially correlated effect to account for the system behaviour not accounted for by $\boldsymbol{e}(\boldsymbol{s}_i, \boldsymbol{b})$ and $\boldsymbol{\epsilon}_i$ is a random, uncorrelated effect. The systematic effects $\boldsymbol{e}(\boldsymbol{s}, \boldsymbol{b})$ could be derived from a kinematic model of a CMM, [14], for example. In the calculations in section 7, we assume that $\boldsymbol{e}(\boldsymbol{s}, \boldsymbol{b})$ relates to scale and squareness errors defined by six parameters $\boldsymbol{b}$ [4]. We assume that some calibration of the CMS has been undertaken to determine an estimate of $\boldsymbol{b}$ to correct for these systematic effects so that their expected value is zero but the uncertainty associated with the estimate of $\boldsymbol{b}$ contributes to the uncertainty associated with $\boldsymbol{x}_i$. This uncertainty contribution is specified by the variance matrix of the form $G_i G_i^{\mathrm{T}}$ where G_i is derived from the $3 \times p$ matrix of sensitivities of $\partial \boldsymbol{e}(\boldsymbol{s}_i, \boldsymbol{b})/\partial b_j$ with respect to $\boldsymbol{b}$. Since, in general, we do not know $\boldsymbol{s}_i$, we evaluate these partial derivatives at $\boldsymbol{e}(\boldsymbol{x}_i, \boldsymbol{b})$ instead. For a vector $\boldsymbol{x}_I$ of measured coordinates, we let $G(\boldsymbol{x}_I)$ be the $3m \times p$ matrix of sensitivities. For $3m \times 1$ and $3m' \times 1$ vectors $\boldsymbol{x}_I$ and $\boldsymbol{y}_{I'}$, we define the $3m \times 3m'$ matrix $H(\boldsymbol{x}_I, \boldsymbol{y}_{I'})$ by

$$H(\boldsymbol{x}_I, \boldsymbol{y}_{I'}) = G(\boldsymbol{x}_I) G^{\mathrm{T}}(\boldsymbol{y}_{I'}). \tag{5}$$

This matrix represents the covariance associated with sets of coordinate data $\boldsymbol{x}_I$ and $\boldsymbol{y}_{I'}$ due to the common systematic effects $\boldsymbol{b}$. With this notation, the variance contribution of $\boldsymbol{b}$ to $\boldsymbol{x}_I$ is $H(\boldsymbol{x}_I, \boldsymbol{x}_I)$.

The effects $\boldsymbol{f}_i = (f_i, g_i, h_i)^{\mathrm{T}}$ are used to account for systematic behaviour of the CMS not modelled by $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{b})$ and thus compensates for any inadequacy in this model [10]. We assume that $\boldsymbol{f}_i$ and $\boldsymbol{f}_q$ are spatially correlated [11, 12] according to

$$\mathrm{cov}(\boldsymbol{f}_i, \boldsymbol{f}_q) = K(\boldsymbol{x}_i, \boldsymbol{x}_q) = \begin{bmatrix} k(\boldsymbol{x}_i, \boldsymbol{x}_q) & & \\ & k(\boldsymbol{x}_i, \boldsymbol{x}_q) & \\ & & k(\boldsymbol{x}_i, \boldsymbol{x}_q) \end{bmatrix}, \tag{6}$$

determined using a correlation kernel k, e.g.,

$$k(\boldsymbol{x}, \boldsymbol{x}') = \sigma_F^2 \exp\left\{ -\frac{1}{2} (\boldsymbol{x} - \boldsymbol{x}')^{\mathrm{T}} M^{-1} (\boldsymbol{x} - \boldsymbol{x}') \right\}, \tag{7}$$

where the 3×3 matrix M defines the scale over which the correlation operates. For isotropic correlation $M = \lambda^2 I$, for some length scale parameter λ. The spatial correlation ensures that $\boldsymbol{f}$ will be similar to $\boldsymbol{f}'$ if $\boldsymbol{x}$ is close to $\boldsymbol{x}'$ where the measure of closeness is determined by M. For $3m \times 1$ and $3m' \times 1$ vectors $\boldsymbol{x}_I$ and

$\boldsymbol{y}_{I'}$, we define $K(\boldsymbol{x}_I, \boldsymbol{y}_{I'})$ to be the $3m \times 3m'$ matrix composed of the 3×3 submatrices $K(\boldsymbol{x}_i, \boldsymbol{y}_q)$. This matrix represents the covariance associated with sets of coordinate data $\boldsymbol{x}_I$ and $\boldsymbol{y}_{I'}$ due to the spatially correlated effects. If $\boldsymbol{y}_q \approx \boldsymbol{x}_i$ then, for the kernel $k(\boldsymbol{x}, \boldsymbol{x}')$ in (6), $K(\boldsymbol{x}_i, \boldsymbol{y}_q) \approx \sigma_F^2 I$. With this notation, the variance contribution of $\boldsymbol{f}_I$ to $\boldsymbol{x}_I$ is $K(\boldsymbol{x}_I, \boldsymbol{x}_I)$.

Finally, the variance contribution of the random effects $\boldsymbol{\epsilon}$ is given by $\sigma^2 I$, where I denotes the $3m \times 3m$ identity matrix. With these definitions,

$$V(\boldsymbol{x}_I, \boldsymbol{x}_I) = \sigma^2 I + H(\boldsymbol{x}_I, \boldsymbol{x}_I) + K(\boldsymbol{x}_I, \boldsymbol{x}_I), \tag{8}$$

$$V(\boldsymbol{x}_I, \boldsymbol{y}_{I'}) = H(\boldsymbol{x}_I, \boldsymbol{y}_{I'}) + K(\boldsymbol{x}_I, \boldsymbol{y}_{I'}), \tag{9}$$

represent the variance matrix associated with $\boldsymbol{x}_I$ and the covariance matrix of $\boldsymbol{x}_I$ with $\boldsymbol{y}_{I'}$ for coordinates measured by the same measuring instrument, respectively.

4 Propagation of point coordinate uncertainties to geometric feature parameters

Given a set of point coordinates $\boldsymbol{x}_I$ representing a geometric element, such as a cylinder, parametrized by parameters $\boldsymbol{a} = (a_1, \ldots, a_n)^{\mathrm{T}}$, the least squares best-fit element [3, 5] is found by solving

$$\min_{\boldsymbol{a}} \sum_{i=1}^{m} d^2(\boldsymbol{x}_i, \boldsymbol{a}),$$

where $d(\boldsymbol{x}, \boldsymbol{a})$ is the orthogonal distance of $\boldsymbol{x}_i$ from the geometric element specified by $\boldsymbol{a}$. Such an optimisation problem can be solved using the Gauss-Newton algorithm [6], for example, and involves the calculation of the $m \times n$ Jacobian matrix J of partial derivatives $J_{ij} = \partial d(\boldsymbol{x}_i, \boldsymbol{a})/\partial a_j$ of d with respect to the parameters $\boldsymbol{a}$. Let J be the Jacobian matrix evaluated at the solution decomposed using the QR factorisation [7] as $J = Q_1 R_1$ where Q_1 is an $m \times n$ orthogonal matrix and R_1 is an $n \times n$ upper triangular matrix. Let N be the $m \times 3m$ block diagonal matrix with $\boldsymbol{n}_i^{\mathrm{T}}$ in the ith row and in columns $3i - 2$ to $3i$, where $\boldsymbol{n}_i$ is the normal vector to the geometric element's surface at the point on the surface closest to $\boldsymbol{x}_i$. The uncertainties associated with the point coordinates, as represented by the variance matrix $V(\boldsymbol{x}_I, \boldsymbol{x}_I)$ given in (8), are propagated through to the uncertainties associated with the fitted parameters [13], represented by the $n \times n$ variance matrix $V_{\boldsymbol{a}}$ according to

$$\begin{aligned} V_{\boldsymbol{a}} &= S_{\boldsymbol{a}} V(\boldsymbol{x}_I, \boldsymbol{x}_I) S_{\boldsymbol{a}}^{\mathrm{T}}, \quad S_{\boldsymbol{a}} = (J^{\mathrm{T}} J)^{-1} J^{\mathrm{T}} N = R_1^{-1} Q_1^{\mathrm{T}} N, \\ &= \sigma^2 (R_1^{\mathrm{T}} R_1)^{-1} + S_{\boldsymbol{a}} H(\boldsymbol{x}_I, \boldsymbol{x}_I) S_{\boldsymbol{a}}^{\mathrm{T}} + S_{\boldsymbol{a}} K(\boldsymbol{x}_I, \boldsymbol{x}_I) S_{\boldsymbol{a}}^{\mathrm{T}}, \end{aligned} \tag{10}$$

involving the $n \times 3m$ matrix $S_{\boldsymbol{a}}$ representing the sensitivity of $\boldsymbol{a}$ with respect to $\boldsymbol{x}_I$. The first term on the right is the variance component due to the random effects $\boldsymbol{\epsilon}_i$, the second, that due to the systematic error parameters $\boldsymbol{b}$ and the third, that due to the correlated systematic effects $\boldsymbol{f}_i$.

5 Uncertainty associated with comparator measurements

Let $\boldsymbol{x}_I^*$ represents measurements of a master artefact and $\boldsymbol{x}_I$ those of a test artefact by the same CMS using nominally the same measurement strategy. If $\boldsymbol{a}^*$ and $\boldsymbol{a}$ are the fitted parameters associated with $\boldsymbol{x}_I^*$ and $\boldsymbol{x}_I$, respectively, and $S_{\boldsymbol{a}^*}$ and $S_{\boldsymbol{a}}$ are the corresponding sensitivity matrices, then the variance matrix $V_{(\boldsymbol{a}^*-\boldsymbol{a})}$ associated with $\boldsymbol{a}^* - \boldsymbol{a}$ is given by

$$V_{(\boldsymbol{a}^*-\boldsymbol{a})} = \left[\begin{array}{cc} S_{\boldsymbol{a}^*} & -S_{\boldsymbol{a}} \end{array} \right] \left[\begin{array}{cc} V(\boldsymbol{x}_I^*, \boldsymbol{x}_I^*) & V(\boldsymbol{x}_I^*, \boldsymbol{x}_I) \\ V(\boldsymbol{x}_I, \boldsymbol{x}_I^*) & V(\boldsymbol{x}_I, \boldsymbol{x}_I) \end{array} \right] \left[\begin{array}{cc} S_{\boldsymbol{a}^*} & -S_{\boldsymbol{a}} \end{array} \right]^{\mathrm{T}}, \tag{11}$$

the counterpart of (3) for the case of a single measurand. The contribution of the random effects to $V_{(\boldsymbol{a}^*-\boldsymbol{a})}$ is given by $\sigma^2((R_1^*)^{\mathrm{T}} R_1^*)^{-1} + \sigma^2(R_1^{\mathrm{T}} R_1)^{-1}$, and that from the systematic errors $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{b})$ is given by

$$\begin{array}{c} S_{\boldsymbol{a}^*} H(\boldsymbol{x}_I^*, \boldsymbol{x}_I^*) S_{\boldsymbol{a}^*}^{\mathrm{T}} - S_{\boldsymbol{a}^*} H(\boldsymbol{x}_I^*, \boldsymbol{x}_I) S_{\boldsymbol{a}}^{\mathrm{T}} + \\ S_{\boldsymbol{a}} H(\boldsymbol{x}_I, \boldsymbol{x}_I) S_{\boldsymbol{a}}^{\mathrm{T}} \;-\; S_{\boldsymbol{a}} H(\boldsymbol{x}_I, \boldsymbol{x}_I^*) S_{\boldsymbol{a}^*}^{\mathrm{T}} \end{array}$$

Since $\boldsymbol{x}_I^*$ is close to $\boldsymbol{x}_I$, we expect $S_{\boldsymbol{a}^*}$ to be close to $S_{\boldsymbol{a}}$ and that all four H matrices in the above to be similar to each other. Thus, the uncertainty contribution from $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{b})$ to $V_{(\boldsymbol{a}^*-\boldsymbol{a})}$ is substantially cancelled out. By the same argument, the uncertainty contributions from the systematic effects $\boldsymbol{f}_i$ are also largely cancelled out. Hence, the variance matrix associated with $\boldsymbol{a}^* - \boldsymbol{a}$ is such that

$$V_{(\boldsymbol{a}^*-\boldsymbol{a})} \approx 2\sigma^2((R_1^*)^{\mathrm{T}} R_1^*)^{-1} \approx 2\sigma^2(R_1^{\mathrm{T}} R_1)^{-1}.$$

6 Collaborative calibration of a test artefact

In the collaborative calibration of a test artefact [8], a CMS, labelled A (for 'absolute' mode), measures the master artefact to produce estimate $\boldsymbol{a}_A^*$ of the feature parameters with associated variance matrix $V_{\boldsymbol{a}_A^*}$. A second CMS, labelled C (for 'comparator' mode), measures the master artefact and the test artefact using nominally the same measurement strategy for both artefacts determining estimates $\boldsymbol{a}_C^*$ and $\boldsymbol{a}_C$ and associated variance matrices $V_{\boldsymbol{a}_C^*}$ and $V_{\boldsymbol{a}_C}$. In addition, the difference $\boldsymbol{a}_C - \boldsymbol{a}_C^*$ is calculated along with the variance matrix $V_{(\boldsymbol{a}_C - \boldsymbol{a}_C^*)} = V_{(\boldsymbol{a}_C^* - \boldsymbol{a}_C)}$ using the approach summarised in (11). The collaborative estimate of $\boldsymbol{a}$, denoted by $\boldsymbol{a}_{AC}$ and associated variance matrix $V_{\boldsymbol{a}_{AC}}$, are given by

$$\boldsymbol{a}_{AC} = \boldsymbol{a}_A^* + (\boldsymbol{a}_C - \boldsymbol{a}_C^*), \quad V_{\boldsymbol{a}_{AC}} = V_{\boldsymbol{a}_A^*} + V_{(\boldsymbol{a}_C - \boldsymbol{a}_C^*)}. \tag{12}$$

If the system A also measures the test artefact, providing an estimate $\boldsymbol{a}_A$ of the parameters and associated variance matrix $V_{\boldsymbol{a}_A}$, the validity of the collaborative estimate $\boldsymbol{a}_{AC}$ can be assessed by evaluating

$$\Delta \boldsymbol{a} = \boldsymbol{a}_A - \boldsymbol{a}_{AC} = \boldsymbol{a}_A - \boldsymbol{a}_A^* - (\boldsymbol{a}_C - \boldsymbol{a}_C^*). \tag{13}$$

If the measurement strategy employed by system A in measuring the test artefact was completely different from that used to measure the master artefact so that the two sets of measurements can be regarded as independent then the variance matrix associated with $\Delta\boldsymbol{a}$ in (13) is given by

$$V^I_{\Delta\boldsymbol{a}} = V_{\boldsymbol{a}_A} + V_{\boldsymbol{a}^*_A} + V_{(\boldsymbol{a}_C - \boldsymbol{a}^*_C)}. \tag{14}$$

If, on the other hand, the system A uses nominally the same measurement strategy (and fixturing, etc.) for measuring the master and test artefact, the variance matrix associated with $\Delta\boldsymbol{a}$ is given by

$$V^C_{\Delta\boldsymbol{a}} = V_{(\boldsymbol{a}_A - \boldsymbol{a}^*_A)} + V_{(\boldsymbol{a}_C - \boldsymbol{a}^*_C)}. \tag{15}$$

The variance matrix $V^C_{\Delta\boldsymbol{a}}$ depends mainly on the random effects associated with systems A and C with the systematic effects associated with both systems largely cancelling out. Note that we do not require that system A and system C use the same measurement strategy, only that each use the same strategy for measuring the master and test artefacts.

7 Numerical example: cylinder feature

We illustrate the methodology for evaluating the uncertainties associated with feature parameter estimates by considering the case of a cylindrical feature. The data consists of 13 nominally uniformly spaced points $\boldsymbol{s}^*_i$ and $\boldsymbol{s}_i$ around two circles at heights 100 mm and -100 mm on a cylinder of radius 100 mm, the $\boldsymbol{s}^*_i$ representing points on the master artefact, the points $\boldsymbol{s}_i$ those on the test artefact. The $\boldsymbol{s}^*_i$ and $\boldsymbol{s}_i$ are derived from points exactly on a cylinder but are perturbed by random effects tangential to the surface and normal to the surface with the tangential and normal effects drawn from the Gaussian distributions $\mathrm{N}(0, \sigma_T^2)$ and $\mathrm{N}(0, \sigma_N^2)$, respectively. These effects simulate the lack of precision of a CMS moving exactly to a pre-assigned point on the surface of an artefact.

The simulated CMS measurements are generated using the model in (4) where the error terms $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{b})$ represent scale and squareness errors

$$\boldsymbol{b} = (b_{xx}, b_{yy}, b_{zz}, b_{xy}, b_{xz}, b_{yz})^{\mathrm{T}},$$

six parameters in total, with $\boldsymbol{b} \sim \mathrm{N}(\boldsymbol{0}, \sigma_B^2 I)$. The correlated effects $\boldsymbol{f}$ are generated according to (6) and (7) with $M = \lambda^2 I$. The random effects ϵ_i are drawn from a normal distribution $\mathrm{N}(0, \sigma^2)$.

The data generated involves simulated measurements by a CMS, labelled A, and two CMSs, labelled C and C', working in comparator mode. The values of the parameters specifying the statistical characterisations of these CMSs are given in table 1. CMS C and particularly C' could be subject to much larger system effects but their repeatability, as represented by σ, is comparable to that of system A. The value of the length scale parameter λ of 5 mm means that if $\boldsymbol{x}$ is 1 mm from $\boldsymbol{x}'$ the uncertainty in $f - f'$ is approximately 1/5th that of the uncertainty in f, so that for $\sigma_F = 0.020$ mm, the uncertainty associated with $f - f'$ is 0.004 mm. While the statistical characterisation for CMS A is appropriate for modelling a reasonably accurate CMM, the statistical characterisation of systems

C and particularly C' represents performances considerably below that of system A. These characterisations have been chosen to demonstrate that comparator mode measurement can be tolerant of (unrealistically) large systematic effects.

The data generated simulated three experiments in which the test artefact has radius $r_0 = 100.0$ mm, the same as the master artefact, $r_0 = 100.2$ mm and $r_0 = 100.5$ mm. As the difference in the radii increases, the cancellation of the systematic effects in comparator mode will be less exact.

The cylinder feature is parametrized by (x_0, y_0) the coordinates of the intersection of its axis with the plane $z = 0$, angles of rotation α and β defining the axis direction vector and its radius r_0. The standard uncertainties, ($k = 1$), associated with the parameters x_0, y_0 and r_0 are given in table 2 and have the following interpretation:

- u_1 Uncertainty $u_A(a_j^*)$ associated with a_j^* determined for CMS A for the master artefact, derived from $V_{\boldsymbol{a}_A^*}$, defined as in (10).
- u_2 Uncertainty $u_A(a_j^* - a_j)$ associated with $a_j^* - a_j$ for CMS A in comparator mode, derived using $V_{(\boldsymbol{a}_A^* - \boldsymbol{a}_A)}$, defined as in (11).
- u_3 Uncertainty $u_C(a_j^*)$ associated with a_j^* determined for CMS C for the master artefact, derived from $V_{\boldsymbol{a}_C^*}$.
- u_4 Uncertainty $u_C(a_j^* - a_j)$ associated with $a_j^* - a_j$ for CMS C in comparator mode, derived using $V_{(\boldsymbol{a}_C^* - \boldsymbol{a}_C)}$.
- u_5 Uncertainty $u_{AC}(a_j)$ associated with a_j for the collaborative calibration of the test artefact derived from $V_{\boldsymbol{a}_{AC}}$, defined as in (12).
- u_6 Uncertainty $u^C(\Delta a_j)$ associated with Δa_j defined in (13) for both CMS A and CMS C used in comparator mode, derived from $V^C_{\Delta \boldsymbol{a}}$, defined as in (15).
- u_7 Uncertainty $u^C(\Delta a_j)$ associated with Δa_j defined in (13) where only CMS C used in comparator mode, derived from $V^I_{\Delta \boldsymbol{a}}$, defined in (14).

The uncertainties in columns labelled u_3' to u_7' are defined in the same way as those in columns labelled u_3 to u_7 only for CMS C'.

Referring to table 2, columns labelled u_1, u_3 and u_3' are the uncertainties associated with the master artefact for CMSs A, C and C'. The potentially large systematic effects associated with systems C and C' lead to large uncertainties associated with the fitted parameters. Columns labelled u_2, u_4 and u_4' are the uncertainties for the systems in comparator mode. Here the performance of systems C and C' are much more comparable with those of system A, as they reflect mainly the random component of the effects. We note that $\sigma = 0.000\,5$ mm for system A and $\sigma = 0.001$ mm for systems C and C'. There is some increase in the uncertainties associated with systems C and C' for experiment 2, middle three rows, and experiment 3, last three rows, due to the increasing difference between the radii of the master and test artefact. Columns labelled u_5 and u_5' are the uncertainties associated with the collaborative calibration of the test artefact and show that the major contribution is from the calibration of the master artefact by system A. The columns labelled u_6 and u_6' show that an accurate validation of the collaborative calibration can be made if A is used in comparator mode, while columns u_7 and u_7' show that the validation is much less accurate if the validation involves the independent measurements of the test artefact using system A. The uncertainties associated with the angle parameters α and β exhibit a similar behaviour.

	A	C	C'
σ_B/(mm/mm)	10^{-5}	10^{-4}	10^{-3}
σ_T/mm	0.002 0	0.005 0	0.010 0
σ_N/mm	0.000 1	0.000 1	0.000 1
σ_F/mm	0.002 0	0.010 0	0.020 0
λ/mm	5.000 0	5.000 0	5.000 0
σ/mm	0.000 5	0.001 0	0.001 0

Table 1: Values of statistical parameters used in the generation of the simulated measurements for systems A, C and C'.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u'_3	u'_4	u'_5	u'_6	u'_7
x_0	0.57	0.20	2.79	0.39	0.69	0.44	0.90	5.55	0.39	0.69	0.44	0.90
y_0	0.57	0.20	2.79	0.39	0.69	0.44	0.90	5.55	0.39	0.69	0.44	0.90
r_0	0.81	0.14	7.34	0.28	0.86	0.31	1.18	70.82	0.28	0.86	0.31	1.18
x_0	0.57	0.20	2.79	0.41	0.70	0.45	0.91	5.55	0.45	0.73	0.49	0.93
y_0	0.57	0.20	2.79	0.41	0.70	0.45	0.91	5.55	0.45	0.73	0.49	0.93
r_0	0.81	0.14	7.34	0.29	0.86	0.32	1.19	70.82	0.35	0.89	0.38	1.20
x_0	0.57	0.20	2.79	0.48	0.75	0.52	0.94	5.55	0.68	0.89	0.71	1.06
y_0	0.57	0.20	2.79	0.48	0.75	0.52	0.94	5.55	0.68	0.89	0.71	1.06
r_0	0.81	0.14	7.34	0.34	0.88	0.37	1.20	70.82	0.60	1.01	0.61	1.30

Table 2: Uncertainties associated with cylinder axis coordinates (x_0, y_0) and radius r_0 for simulated data for three experiments. All units are micrometres. See the main text for details.

8 Concluding remarks

This paper has presented a general model for the uncertainty contributions associated with coordinate measuring systems arising from systematic and random effects. The systematic effects can be described in terms of known error models or in terms of spatially correlated effects or both. The model can be used to determine the covariance associated with two sets of coordinate data gathered by the same system, for example, data associated with a master and test artefact. In particular, the methodology addresses the issue of how the influence of systematic effects associated with the measurement systems are much reduced in comparator mode and enables the accurate collaborative calibration of test artefacts using a master artefact calibrated by a high-accuracy system along with comparator measurements of the master and test artefacts using a second system. The uncertainty associated with the collaborative calibration depends mainly on the absolute performance of the system calibrating the master artefact and on the repeatability of the system operating in comparator mode.

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