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Development of a Critical Assembly Quality Analysis Method for Machine Tools Based on the Jacobian-Torsor Model

Lin Tsao[1], Sheng-Hsiang Hung[2]

[1,2] Center for Measurement Standards, Industrial Technology Research Institute, Hsinchu, Taiwan

Abstract
To enhance the precision and reliability of machine tools, assembly quality has become a critical concern in high-precision manufacturing. This study proposes a systematic assembly quality analysis method based on the Jacobian–Torsor model. By integrating geometric specifications, tolerance information, and the kinematic chain of mechanical components, an error propagation model is established to evaluate positioning accuracy across six degrees of freedom. Monte Carlo simulation is employed to statistically quantify tolerance contributions and identify critical assembly parameters. The results demonstrate that dominant tolerances can be clearly ranked using contribution and sensitivity indices, enabling targeted optimization during design and assembly stages. The proposed framework provides a digital foundation for intelligent tolerance allocation and precision enhancement in advanced machine tool manufacturing.

Jacobian-Torsor model, Small displacement torsor, Geometric error

1. Introduction

Assembly quality analysis and precision compensation techniques for machine tools are currently developed independently by individual manufacturers, lacking a unified analytical framework. Consequently, improvements in machining accuracy often rely on repeated trial-and-error processes on physical machines, resulting in extended occupation time, reduced productivity, and increased resource consumption.

Under the digital twin paradigm, the integration of virtual modeling and data-driven analysis provides new opportunities for systematic precision optimization. However, a structured method to model assembly error propagation and quantitatively evaluate tolerance contributions remains insufficiently developed.

This study proposes a digital modeling approach for analyzing critical assembly parameters of linear axis systems in multi-axis machine tools. Based on the Jacobian–Torsor framework, dimensional and assembly errors are propagated through the kinematic chain according to the assembly sequence. The model takes key assembly tolerances as inputs and outputs six geometric error components (three translational and three rotational). By ranking tolerance contributions, the method enables precise identification of dominant assembly parameters and supports targeted quality improvement.

Tolerance analysis methodologies based on the Jacobian–Torsor model have been widely studied [1–3], while geometric error compensation in machine tools has been extensively addressed in precision engineering research [6–8].

2. Jabonian-Torsor model

The Jacobian–Torsor Model [1,2] is a theoretical framework that integrates the Small Displacement Torsor (SDT) with the Jacobian matrix, and is widely applied in tolerance analysis and geometric error propagation modeling of assemblies. In this approach, SDTs are used to represent small deviations between actual geometric features and their nominal ideal positions. The Jacobian matrix then maps these local geometric errors to the global assembly system, enabling the propagation and accumulation of errors across multiple components. Owing to its capability to simultaneously evaluate geometric sensitivity and support statistical analysis, the Jacobian–Torsor Model is particularly well suited for comprehensive assembly error assessment and process tolerance optimization in complex mechanical systems. It provides an effective and systematic analytical tool for quality control and predictive modeling in precision mechanical assembly.[2,5]

2.1. Small displacement torsor

The Small Displacement Torsor (SDT) is used to represent the infinitesimal motion of a rigid body in space. It consists of six motion components, including three translational components (u, v, w) and three rotational components (α, β, γ), as shown in **Equation (1)**.

$$\mathbf{T}_{\mathrm{i}} = \{\mathrm{u}_{\mathrm{i}} \;\; \mathrm{v}_{\mathrm{i}} \;\; w_{\mathrm{i}} \;\; \alpha_{\mathrm{i}} \;\; \beta_{\mathrm{i}} \;\; \gamma_{\mathrm{i}}\}^{\mathrm{T}} \tag{1}$$

This six-dimensional vector provides a complete description of the small deviations of a geometric feature from its nominal ideal position. In practical engineering applications, most geometric features of mechanical components can be classified into seven basic geometric types, such as planes, lines, and cylinders. Different geometric features correspond to different tolerance requirements, including flatness, straightness, and cylindricity, which in turn lead to different forms of SDT representations.

Taking a planar feature as an example, the SDT modeling and constraint formulation are illustrated. In theory, a plane possesses six degrees of freedom. However, under practical geometric tolerances and functional constraints, its degrees of freedom are restricted by positioning and assembly conditions. As shown in Fig. 1, only the translational displacement along the z-axis and the rotational motions about the x and y axes are permitted. Consequently, the SDT representation of the planar feature can be expressed as $(0, 0, w, \alpha, \beta, 0)$.

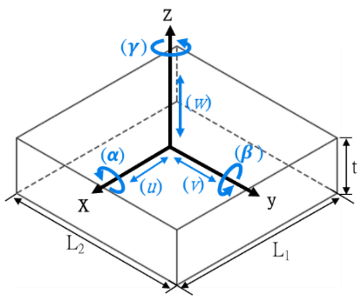

Figure 1. Fundamental geometric characteristics of flatness.

2.2. Jacobian matrix

In the Jacobian–Torsor model, the Jacobian matrix is used to describe how the Small Displacement Torsors (SDTs) of individual Functional Elements (FEs) are propagated to the Functional Requirement (FR). The Jacobian matrix is derived from the coordinate transformations between the local coordinate systems of the FEs and the global coordinate system associated with the functional requirement. Each functional element corresponds to a 6×6 submatrix, which represents its contribution and influence on the FR.

$$[J]_{FE_i} = \begin{bmatrix} [R_0^i]_{3\times3} & [W_i^n]_{3\times3}\cdot[R_0^i]_{3\times3} \\ 0 & [R_0^i]_{3\times3} \end{bmatrix}_{6\times6} \quad (2)$$

where $\mathbf{W}_i^n \in \mathbb{R}^{3\times3}$ is the skew-symmetric matrix of the position vector, representing the effect of rotational motion on translational displacement. $\mathbf{R}_0^i \in \mathbb{R}^{3\times3}$ is the rotation matrix of the functional element with respect to the global coordinate system.

The skew-symmetric matrix $\mathbf{W}_i^n$ is defined as:

$$\mathbf{W}_i^n = \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \quad (3)$$

where $(x_i,\ y_i,\ z_i)$ denotes the position vector of the functional element. The block structure of Eq. (2) reflects the rigid-body kinematic mapping between the rotational and translational components of the SDT.

2.3. Calculation Method and Steps

The primary sources of the final precision target error stem from manufacturing and assembly deviations in individual components, represented by torsor $T_1, T_2, \ldots, T_n$ and corresponding Jacobian matrices $J_1, J_2, \ldots, J_n$. The total error $[J]_{FR}$ can be expressed as:

$$[J]_{FR} = \sum_{i=1}^{n}[J]_{FE_i}\cdot T_{FE_i} \quad (4)$$

Since it is known that dimensional/assembly errors in mass production generally follow specific probability distributions (e.g., normal distribution, rectangular distribution), statistical Monte Carlo simulation is applied to this model to accurately reflect the impact of error sources on overall precision. By randomly sampling error sources under this specific probability distribution and generating large datasets through iterative methods for simulation analysis.

3. Linear Axis Structural Assembly Analysis

In linear-axis-dominated error propagation paths, the geometric characteristics of individual components are closely associated with their assembly relationships. The guide system considered in this study—including the base, linear guide rails, slide blocks, and workbench—was characterized in terms of nominal dimensions, permissible tolerance ranges, and potential geometric deviation types (e.g., flatness, straightness, and parallelism). These specifications are summarized in Table 1.

Based on the assembly hierarchy, a small-displacement torsor (SDT) model was established to represent infinitesimal geometric deviations. The corresponding Jacobian matrix was constructed according to the error propagation sequence (base → guide → slider → workbench), as illustrated in Fig. 2. [3]

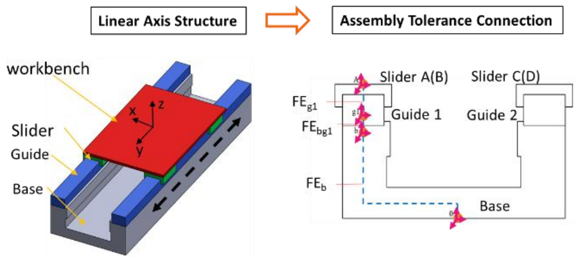

Figure 2. Configuration of the linear axis workbench and corresponding geometric error chain based on the Jacobian–Torsor model.

Furthermore, Monte Carlo simulation was employed to generate random samples representing stochastic geometric deviations that may occur during the actual assembly process. Six-degree-of-freedom geometric errors were simulated for each functional component according to the specified tolerance distributions.

The assembly errors of individual linear axes were first evaluated independently. These errors were then superimposed according to the structural configuration of the machine tool to obtain the overall system-level geometric deviation.

Subsequently, statistical sensitivity analysis and tolerance contribution evaluation were conducted using the control variable method to quantify the influence of each assembly quality parameter on the total geometric error. The dimensional specifications and tolerance ranges of all components used in the simulation are summarized in Table 1.

Table 1 . Geometric parameters and tolerance ranges used in the simulation.

Component	dimension(mm)	Tolerance Items	Tolerance
Base	1200×800×50 ($L_b \times W_b \times H_b$)	Flatness f_b	0.01 mm
Guide 1	800×45×38 ($L_{g1} \times W_{g1} \times H_{g1}$)	Flatness f_{g1}	0.01 mm
		straightness S_{M1}	0.01 mm/m
		Installation straightness in the x-direction S_{1x}	0.01 mm/m
		Installation straightness in the z-direction S_{1z}	0.01 mm/m
Guide 2	800×45×38 ($L_{g2} \times W_{g1} \times H_{g2}$)	Flatness f_{g2}	0.01 mm
		Straightness S_{M2}	0.01 mm/m
		Installation straightness in the x-direction S_{2x}	0.01 mm/m
		Installation straightness in the z-direction S_{2z}	0.01 mm/m

3.1. Analysis of Assembly Quality Control Parameters

Applying the Jacobian-Torsor model, the six geometric errors of a slider A in the calculation are expressed as:

$$\begin{bmatrix} u \\ v \\ w \\ \alpha \\ \beta \\ \gamma \end{bmatrix}_{FR_A} = \boldsymbol{J}\times\boldsymbol{D} = [\boldsymbol{J_b} \quad \boldsymbol{J_{bg1}} \quad \boldsymbol{J_{g1}}]\begin{bmatrix} \boldsymbol{T_b} \\ \boldsymbol{T_{bg1}} \\ \boldsymbol{T_{g1}} \end{bmatrix} \quad (5)$$

The workbench is supported by four sliders (A, B, C, and D) and a lead screw. Each individual error chain is propagated through the assembly process and integrated into a total system-level error chain. The overall translational error of the workbench—resulting from the superposition of errors in the same direction across the four sliders—can be expressed as:

$$\begin{bmatrix} u_O \\ v_O \\ w_O \end{bmatrix} = \frac{1}{4}\left\{\begin{bmatrix} u_A \\ v_A \\ w_A \end{bmatrix} + \begin{bmatrix} u_B \\ v_B \\ w_B \end{bmatrix} + \begin{bmatrix} u_C \\ v_C \\ w_C \end{bmatrix} + \begin{bmatrix} u_D \\ v_D \\ w_D \end{bmatrix}\right\} \quad (6)$$

The rotational deviations of the workbench around the X, Y, and Z axes (pitch, yaw, and roll) are determined by the relative vertical and lateral displacements of the four sliders. These relationships can be expressed as:

$$\begin{bmatrix}\alpha_o\\ \beta_o\\ \gamma_o\end{bmatrix}=\begin{bmatrix}\dfrac{\frac{w_B+w_D}{2}-\frac{w_A+w_C}{2}}{D_y}\\ \dfrac{\frac{w_C+w_D}{2}-\frac{w_A+w_B}{2}}{D_x}\\ \frac{1}{4}\left(\frac{u_A-u_B}{D_y}+\frac{v_C-v_A}{D_x}+\frac{u_C-u_D}{D_y}+\frac{v_D-v_B}{D_x}\right)\end{bmatrix} \tag{7}$$

Where D_x and D_y denote the distances between sliders along the X and Y direction respectively.

Using the Monte Carlo simulation approach to statistically analyze six-degree-of-freedom geometric errors. The error propagation mechanism is established based on the Jacobian–Torsor model, incorporating the statistical distributions of typical geometric error sources from individual components through random sampling.

Analysis points are defined at 5 mm intervals along the entire travel range of the linear axis. At each position, 10,000 random samples are generated to perform numerical simulations and evaluate the resulting error distributions. Finally, the variation range of the six geometric errors is quantified using the 3σ confidence interval to characterize the statistical behavior of the system's spatial accuracy.

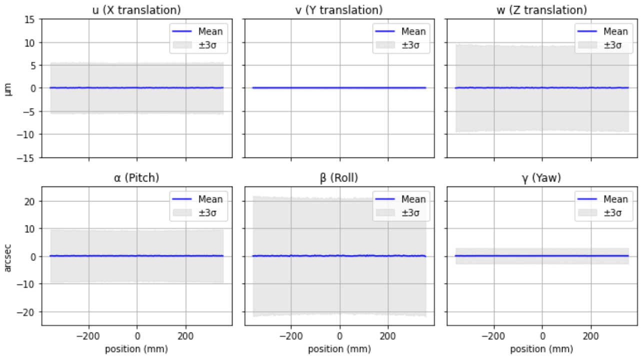

Figure 3. Distribution of six-degree-of-freedom geometric errors along the linear axis travel range.

3.2. Critical Tolerance Analysis of Assembly Tolerances

Critical tolerance identification was performed based on two quantitative metrics:

(1) percentage contribution of each tolerance source and

(2) tolerance sensitivity.

Assuming that all geometric error sources are statistically independent and follow random distributions (e.g., normal distributions), the standard deviation of the overall platform error can be obtained by variance superposition:

$$\sigma_{total}=\sqrt{\sum_i \sigma_i^2} \tag{8}$$

where σ_i denotes the standard deviation of the error induced by the i-th tolerance source. The influence of each error source on the final error can be expressed as $FE_i = J_i \times T_i$, where FE_i represents the impact of the i^{th} error term on the output error, J_i denotes the corresponding Jacobian matrix, and T_i is the small displacement vector of that error source. The contribution percentage of each error source to the total variation is further calculated, where σ is the standard deviation.

Tolerance sensitivity quantifies the extent to which variation in an individual tolerance source affects the overall assembly error.Sensitivity was evaluated using the one-at-a-time (OAT) method, in which:

- only one tolerance source is randomly varied
- all other tolerances are set to zero

Monte Carlo simulations are then performed to observe the resulting variation in the six-degree-of-freedom output. The standard deviation (or variance) of the response is used as the sensitivity index, $S_{i,j}$, representing the sensitivity of tolerance i to degree of freedom j.

To jointly evaluate both statistical contribution and sensitivity, a critical tolerance index is defined as:

$$k_{i,j}=(\%PC)_{i,j}\times S_{i,j} \tag{8}$$

This index reflects both:

- how much a tolerance contributes to overall variation
- how strongly the system responds to that tolerance

For each degree of freedom *j*, tolerance sources are ranked in descending order of $k_{i,j}$ to identify the dominant contributors to assembly error. This ranking provides a quantitative basis for targeted tolerance optimization, enabling precision improvement without uniformly tightening all tolerances.

4. Results

Monte Carlo simulation indicates that the maximum 3σ translational deviation in the u-direction reaches approximately 5 μm at the center of the linear axis travel. As shown in Fig. 4, contribution analysis shows that tolerances S_{1x} and S_{2x} are the dominant contributors in this direction, jointly accounting for more than 80% of the total error variance.

For the v- and w-directions, base plate flatness is identified as the primary influencing factor. It exhibits both the highest percentage contribution (Fig. 4) and the largest sensitivity index (Fig. 5). The remaining tolerance parameters each contribute less than 10% individually, indicating relatively minor influence on overall geometric deviation.

Furthermore, the dominance of tolerance sources varies across different degrees of freedom. As illustrated in Fig. 6, the critical tolerance ranking differs for each directional error component, confirming that geometric error behavior is direction-dependent rather than uniformly distributed.

These results demonstrate that volumetric accuracy is governed by a limited subset of assembly tolerances. Therefore, targeted optimization of dominant tolerances can significantly improve positioning performance without uniformly tightening all tolerance specifications. The results quantitatively confirm that geometric error propagation in linear axis assemblies is anisotropic and governed by structurally coupled tolerance parameters.

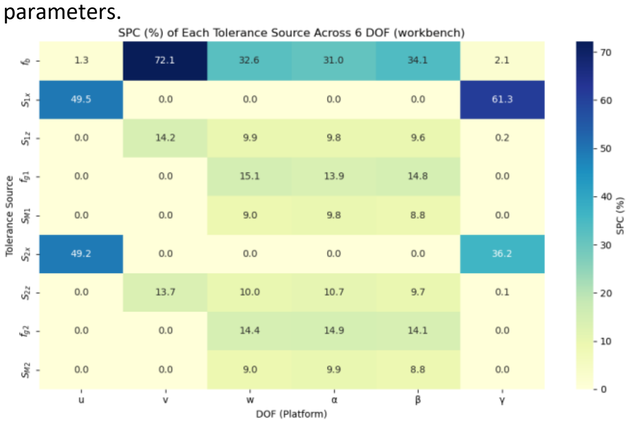

Figure 4. Percentage contribution of individual tolerance sources to the geometric error of the linear axis workbench.

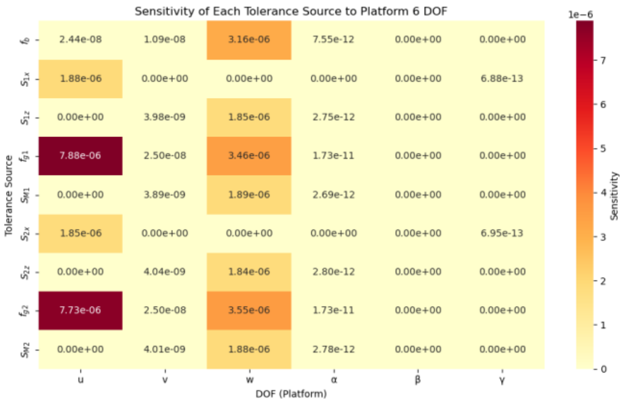

Figure 5. Sensitivity analysis of tolerance parameters affecting workbench geometric deviations.

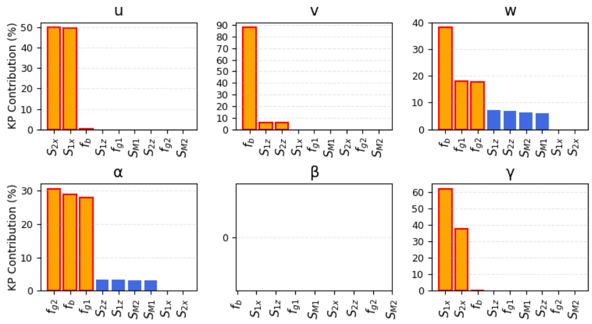

Figure 6. Identified critical tolerances for each degree of freedom at the center position of the linear axis workbench.

5. Summary

This study developed a Jacobian–Torsor-based tolerance analysis framework to quantitatively identify critical assembly parameters in linear axis systems. The results demonstrate that volumetric error is predominantly governed by a limited subset of tolerance sources, with the top three parameters accounting for more than 80% of total error variance. These findings indicate that uniformly tightening all tolerances is neither necessary nor efficient.

Instead, targeted optimization of dominant tolerances can significantly enhance positioning accuracy while reducing assembly cost and complexity. The proposed framework provides a systematic and computationally efficient foundation for precision-oriented assembly design, enabling the transition from empirical adjustment to data-driven tolerance allocation.

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