euspen's 25th International Conference & Exhibition, Zaragoza, ES, June 2025
www.euspen.eu

# Parallel implicit sparse identification of nonlinear dynamics for a 2DOF system

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**Abstract**
Data-driven methods have emerged as a pivotal area of system identification and among these the SINDy-PI (parallel implicit sparse identification of nonlinear dynamics) algorithm stands out for its ability to identify underlying parameters in equations for dynamic systems from measurement data. This method benefits from the capacity to deal with sparse nonlinear dynamical systems and robustness to noisy data. In this work, a dynamic model for a 2 Degree of Freedom, 5 bar linkage, system with two actuated inputs based on equations is developed in Matlab; by using SINDy-PI, unknown parameters of this model are estimated from simulated data with low RMSE(root mean square error) of 0.0501 to simulated results; The case demonstrates the feasibility of applying SINDy-PI on a 2DoF mechanical system and results indicate substantial improvements in noice robustness.

Nonlinear system identification, SINDy-PI, robotics, simulation

## 1. Introduction

In the rapidly evolving field of robotics, understanding and predicting the dynamic behaviour of complex mechanical systems presents a significant challenge. To date, contributions have been done in the study of robot dynamics with several ideas proposed to simplify dynamic models using numerical or symbolical approaches such as applying least square method to estimate dynamic parameters [1-2]. These methods mostly lead to reduced or approximate dynamic equations, but for industrial or research purposes, an accurate and comprehensive dynamic model can provide a better precision in prediction, estimation and control. Many attempts have been made to autonomously extract physical laws from data and with the advancement of machine learning, this approach has become more feasible [3]. Methods involve extracting governing equations from either: (1) probabilistic machine learning (Gaussian process) [4], (2) physics-informed neural network framework [5], (3) physics-informed generative adversarial networks [6]. However, one of the issues is that in most cases, machine-learning methods are black boxes that result in a lack of traceable insight into how the equations are derived and used to obtain results.

In a recent work from Brunton et. al. [7-8], a data-driven method, sparse identification of nonlinear dynamics (SINDy), was proposed to generate nonlinear dynamics from the measurement data. SINDy requires a certain level of prior knowledge regarding the specified system, upon which the reconstructed formulas are predicted. This approach not only inherits the robustness and stability characteristic of the traditional least square method, but also embodies a degree of flexibility associated with machine learning methods.

SINDy has emerged as an effective approach to find the underlying structure of dynamics and therefore, many variants were proposed to broaden its applications: SINDy with control [8], SINDy-PDE [9], Lagrangian-SINDy [10], SINDy-PI [11]. These algorithms extended the original SINDy with partial differential equations, implicit differential equations and differential equations from noise. Among these, SINDy-PI was developed to address the shortcomings of original SINDy in handling rational non-linearity [ref].

This paper takes advantages of the computing power of the SINDy-PI method to develop a novel and efficient parametric identification approach on a two degree of freedom (2DoF) five-bar linkage mechanism. Firstly, the generalised kinematic and dynamic equations of the system are presented; then, based on these equations, an open-loop simulated model is developed using Matlab. Finally, the resulting simulated data is imported into SINDy-PI algorithm to derive dynamic equations and their accuracy evaluated.

## 2. 2-DOF linkage mechanism modelling

The 2DoF five-bar linkage mechanism, shown in **Figure 1**, is a less complex nonlinear system compared to robotic arms, making it a suitable platform for initial validation of the SINDy-PI algorithm. Future work will evaluate the use of SINDy-PI on a 6 DOF compliant robot (a Universal Robot UR5). To develop a simulation the necessary modelling includes basic kinematics and dynamic equations.

### *2.1. Forward and Inverse kinematics*

The simplified geometry of the mechanism developed in this paper is shown in **Figure 1**. Four bars with length $L_1, L_2, L_3, L_4$ are linked by five joints ($L_0$ considered as base) and two torque inputs are located at points *A* and *B* to provide actuation for controlling the movement of endpoint $\boldsymbol{E}$ (with coordinates $E_x$ and $E_y$). $\theta_1, \theta_2, \theta_3, \theta_4$ represents four joint angles with values, respectively. As an initial test, the mass of all four linkages are set to 1 Kg, the length of all four linkages are set to one meter, while the mass and friction from the joints are neglected.

Forward kinematics, which computes the position of end-effector based on the known geometry and joint parameters (angles) is applied As:

$$E_x = L_1 cos\theta_1 + L_2 cos\theta_2$$
$$E_y = L_1 sin\theta_1 + L_2 sin\theta_2 \,. \tag{1}$$

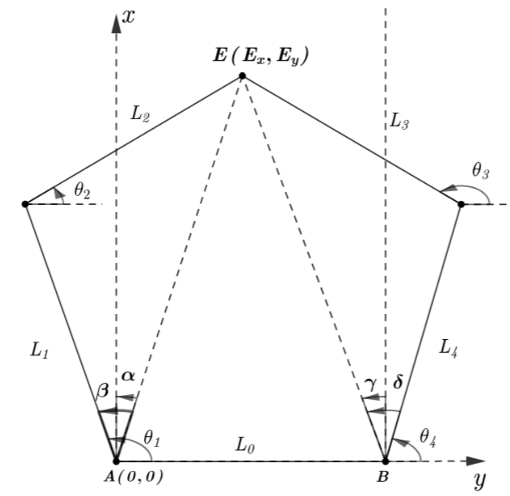


**Figure 1.** Diagram of 2-DOF linkage mechanism

Modifying **Equation** (1) with parameters for the right hand side ($L_3$, $L_4$, $\theta_3$, $\theta_4$) would result in two 2-bar linkages, that do not necessarily connect at point E. To link the two elements together geometric constraints should be considered [12]:

$$f_1 = L_1 cos\theta_1 + L_2 cos\theta_2 - L_3 cos\theta_3 - L_4 cos\theta_4 - L_0 = 0 \tag{2}$$

$$f_2 = sin\theta_1 + L_2 sin\theta_2 - L_3 sin\theta_3 - L_4 sin\theta_4 = 0 \tag{3}$$

Take the second derivative of Equations (2) and (3) with respect to time:

$$\frac{d^2 f_1}{dt^2} = - L_1\ddot{\theta}_1 sin\theta_1 - L_1\dot{\theta}_1^{\,2} cos\theta_1 - L_2\ddot{\theta}_2 sin\theta_2 - L_2\dot{\theta}_2^{\,2} cos\theta_2 + L_3\ddot{\theta}_3 sin\theta_3 + L_3\dot{\theta}_3^{\,2} cos\theta_3 + L_4\ddot{\theta}_4 sin\theta_4 + L_4\dot{\theta}_4^{\,2} cos\theta_4 \tag{4}$$

$$\frac{d^2 f_2}{dt^2} = L_1\ddot{\theta}_1 cos\theta_1 - L_1\dot{\theta}_1^{\,2} sin\theta_1 + L_2\ddot{\theta}_2 cos\theta_2 - L_2\dot{\theta}_2^{\,2} sin\theta_2 - L_3\ddot{\theta}_3 cos\theta_3 + L_3\dot{\theta}_3^{\,2} sin\theta_3 - L_4\ddot{\theta}_4 cos\theta_4 + L_4\dot{\theta}_4^{\,2} sin\theta_4 \,. \tag{5}$$

Since $\theta_2$ and $\theta_3$ are passive joints with no direct torque inputs, they can then be expressed using active angles of the system $\theta_1$ and $\theta_4$. Based on a trigonometry method the relation is:

$$\theta_3 = 2tan^{-1}\left(\frac{A \pm \sqrt{A^2+B^2-C^2}}{B-C}\right) \tag{6}$$

where:

$$A = 2L_3L_4 sin\theta_4 - 2L_1L_3 sin\theta_1$$
$$B = 2L_3L_0 - 2L_1L_3 cos\theta_1 + 2L_3L_4 cos\theta_4$$
$$C = L_1^{\,2} - L_2^{\,2} + L_3^{\,2} + L_3^{\,2} + L_0^{\,2} - L_1L_4 sin\theta_1 sin\theta_4 - 2L_1L_5 cos\theta_1 - 2L_4L_0 cos\theta_4 - 2L_1L_4 cos\theta_1 cos\theta_4$$

$$\theta_2 = sin^{-1}\left(\frac{L_3 sin\theta_3 + L_4 sin\theta_4 - L_1 sin\theta_1}{L_2}\right) \tag{7}$$

### 2.2. Dynamics overview

The standard governing dynamic equation of a system derived from Lagrangian formulation is given as follow:

$$\boldsymbol{M}(\theta)\ddot{\theta} + \boldsymbol{V}(\theta,\dot{\theta}) + \boldsymbol{g}(\theta) = \boldsymbol{u}$$

(8)

where $\boldsymbol{M}$, $\boldsymbol{V}$, $\boldsymbol{g}$ and $\boldsymbol{u}$ denote the inertia, centrifugal and Coriolis force, gravitational force and external force matrix. In the case of this 2DoF system, $\theta, \dot{\theta}$ and $\ddot{\theta}$ refer to joint angle displacement, velocity and acceleration, respectively.

Since the 2DoF system works horizontally, the gravitational force (g) is neglected, leaving just the first two dominating parts $\boldsymbol{M}$ and $\boldsymbol{V}$. Considering that this system has constraints of $\boldsymbol{f} = (f_1, f_2)$ (equation(2) and (3)), this results in the following dynamic equation [13]:

$$\boldsymbol{M}(\theta)\ddot{\theta} + \boldsymbol{V}(\theta,\dot{\theta}) = \boldsymbol{u} + \sum_i \boldsymbol{\lambda}_i \frac{\partial \boldsymbol{f}_i}{\theta_j} \,. \tag{9}$$

$\boldsymbol{\lambda}_i$ ($\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}$) is then a vector for Lagrange multipliers. Combining **Equation** (9) with (4) and (5), six equations with six unknown variables (including joint angles $\theta_1, \theta_2, \theta_3, \theta_4$ and Lagrange multipliers $\lambda_1$ and $\lambda_2$ ) are considered and the equations of motion for 2-DOF linkage mechanism with constraints are computed. Based on these equations, an open loop Matlab simulation is built and data is recorded for use in the SINDy-PI algorithm.

The resulting equations of motion can be found in the following repository:

https://github.com/Minrui-uon/2DOF-system-with-constraints.

## 3. SINDy-PI Overview

The original SINDy algorithm identifies the nonlinear dynamic system equation (6) from measurement data :

$$\frac{d}{dt}x = f(x) \tag{10}$$

$x$ represents the state of system. A series of state $\boldsymbol{X}$ along with related time derivatives $\dot{\boldsymbol{X}}$ can be measured:

$$\boldsymbol{X} = \begin{bmatrix} x^T(t_1) \\ x^T(t_2) \\ \vdots \\ x^T(t_n) \end{bmatrix} \quad \dot{\boldsymbol{X}} = \begin{bmatrix} \dot{x}^T(t_1) \\ \dot{x}^T(t_2) \\ \vdots \\ \dot{x}^T(t_n) \end{bmatrix} \tag{11}$$

### *3.1. Original SINDy Derivation*

The dynamics of the system can be reformed as coefficient vector Ξ multiplied with candidate functions library $\boldsymbol{\Theta}(\boldsymbol{X})$ [7]:

$$\dot{\boldsymbol{X}} = \Xi\, \boldsymbol{\Theta}(\boldsymbol{X}). \tag{12}$$

The sparse representation of the system is then identified through sparse regression methods. It is crucial that a suitable candidate functions library is imported for SINDy to output accurate equations. As it is not always clear what terms are active in a given dynamic system, one of the challenges for applying SINDy is to search for these library functions.

In the case of robotic dynamics with rational non-linearity, it is difficult to deal with massive and complex inverse $\boldsymbol{M}$ and $\boldsymbol{V}$ matrices when transferring the standard dynamic **Equation** (8) into the form of **Equation** (12). Therefore, an extended version named SINDy-PI is applied to avoid matrix transforming issue.

### *3.2 SINDy-PI*

The SINDy-PI (parallel implicit) extends the original SINDy into solving implicit nonlinear equations by making the following change to the general function $f(x)$ [11]:

$$f(x, \dot{x}) = 0 \tag{13}$$

by generalizing the function library $\boldsymbol{\Theta}$ for both $\boldsymbol{X}$ and $\dot{\boldsymbol{X}}$:

$$\boldsymbol{\Theta}(\boldsymbol{X}, \dot{\boldsymbol{X}})\Xi = 0. \tag{14}$$

One of the key features of SINDy-PI is that each candidate function appearing in $\boldsymbol{\Theta}$ will be used to produce separate models, therefore the number of models it searches will grow as the function library grows. This may cause an increase in computing time in the case of complex nonlinear systems.

## 4. Results and discussion

First, the equations proposed in this paper are validated using Simscape Multibody simulation, before being used to generate the candidate function library for SINDy-PI. In the next step, dynamic equations estimated by SINDy-PI are validated.

### *4.1 Equations of motion validation*

To validate dynamic equations with constraints proposed in this paper, a Simscape Multibody model is developed in this section and is considered as our benchmark model ($Model_{Simscape}$). As mentioned in section **2.1**, the mass of all four linkages are set to 1 kg, the length of all four linkages are set to 1 m, while internal spring-damper force law with damping coefficient set to $1e-3\ N.m/(rad/s)$ is applied at the joints to eliminate possible high frequency noise. Sine wave displacement inputs: $\theta(t) = 0.0698sin(t + \pi/2)$ are linked to both fixed joints, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_4$, and actuation torques applied to the joints are recorded. The torque data is then applied on our Simulink model based on equations proposed ($Model_{equation}$) in Section **2.1** to compute joint angle evolution over time and can be used for comparison.

**Figure 1** shows the comparison of $\theta_1, \theta_2, \theta_3,$ and $\theta_4$ between $Model_{Simscape}$ and $Model_{equation}$. As can be seen, for the first 4 s, the error is kept very small. But after that, larger error value is observed. This is most likely because damping elements existing within $Model_{Simscape}$ are not considered within $Model_{equation}$. This results in slightly higher forces that translate to displacement drift in $Model_{equation}$ seen in Figure 2 showing drift in the $E_x$ and $E_y$ direction. Future models for $Model_{equation}$ will seek to include damping elements. The root mean square error (RMSE) is applied here to estimate the accuracy:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(theta_{Simscape_i} - theta_{equation_i})^2} \qquad (15)$$

where:
$n$: number of data/value points;
$theta_{Simscape_i}$ : original joint ata from $Model_{Simscape}$;
$theta_{Simulink_i}$: joint data produced from $Model_{equation}$ .

In summary, $Model_{equation}$ succeeds in reproducing joints movements with RMSE values less than 0.04 rad in all 4 joints.

### *4.2 SINDy-PI recreation of system*

As described in equation (14), the dynamics of the 2DoF mechanism has a state vector $\boldsymbol{X_m} = [\theta_1\ \theta_2\ \theta_3\ \theta_4\ \dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3\ \dot{\theta}_4]$ and derivative vector $\dot{\boldsymbol{X}}_{\boldsymbol{m}} = [\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3\ \dot{\theta}_4\ \ddot{\theta}_1\ \ddot{\theta}_2\ \ddot{\theta}_3\ \ddot{\theta}_4]$ **.** The input torques, $\boldsymbol{u_m} = [\mathrm{u}_1\ \mathrm{u}_2]$, would be the same data gathered in **4.1** applied on joints A and B. Data of all stated variables are collected over 10 seconds at a time step of 0.0001 s to train the SINDy-PI algorithm using $Model_{Simscape}$ . The candidate function library is generated based on equations validated in **4.1**. By taking SINy-PI estimated equations back into the equations used in $Model_{equation}$ resulting in new parameters for $Model_{SINDy-PI}$, the joint data is then reproduced and compared with both data from $Model_{Simscape}$ and $Model_{equation}$. The root mean square error (RMSE) is also applied here to estimate the accuracy of SINDy-PI identified models.

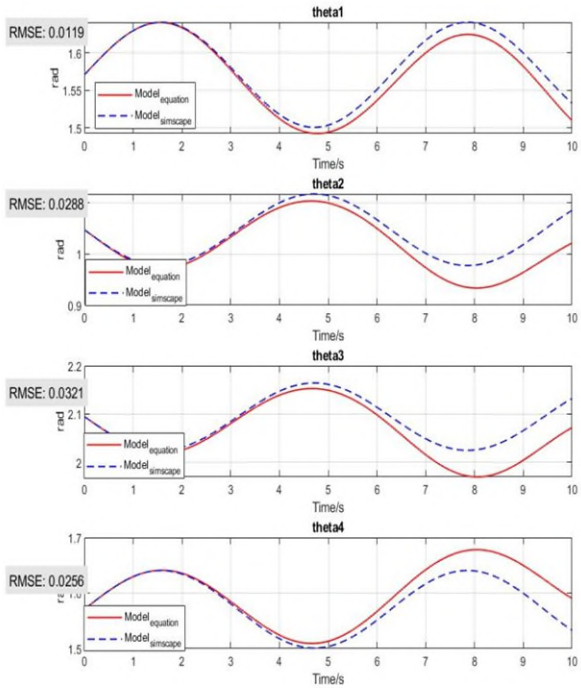


**Figure 1.** Comparison of joint angles between $Model_{Simscape}$ and $Model_{equation}$

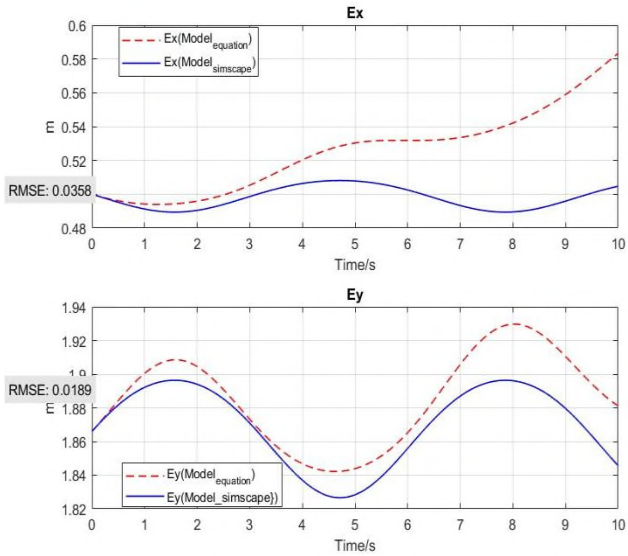


**Figure 2. Comparison of Ex and Ey between $\mathbf{Model_{equation}}$ and $\mathbf{Model_{Simscape}}$**

Based on the collected data, SINDy-PI correctly identified governing equations for all four joints with maximum overall RMSE of 0.0501 rad for four joints (see **Figures 3-6)**.

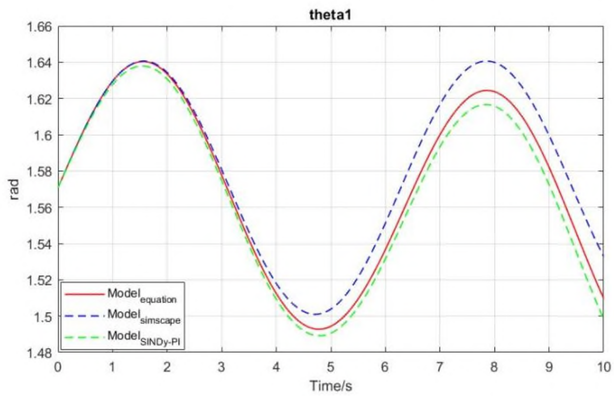

**Figure 3.** Comparison of joint angle $\theta_1$ among $\mathrm{Model}_{\mathrm{equation}}$, $Model_{Simscape}$ and $Model_{SINDy-PI}$

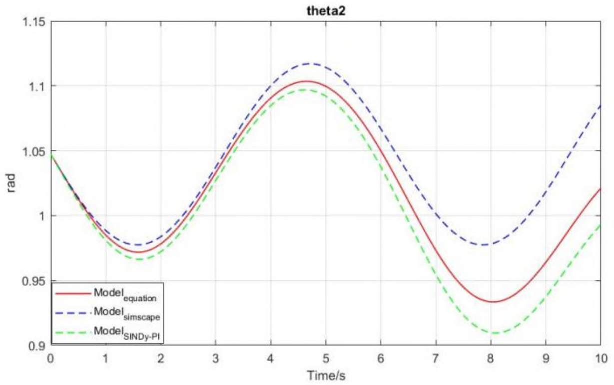


**Figure 4.** Comparison of joint angle $\theta_2$ among $Model_{equation}$, $Model_{Simscape}$ and $Model_{SINDy-PI}$

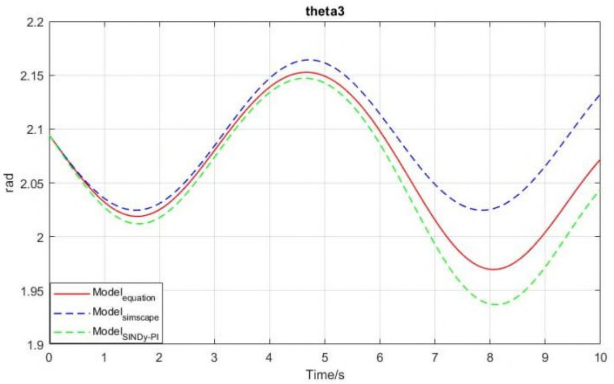


**Figure 5.** Comparison of joint angle $\theta_3$ among $Model_{equation}$, $Model_{Simscape}$ and $Model_{SINDy-PI}$

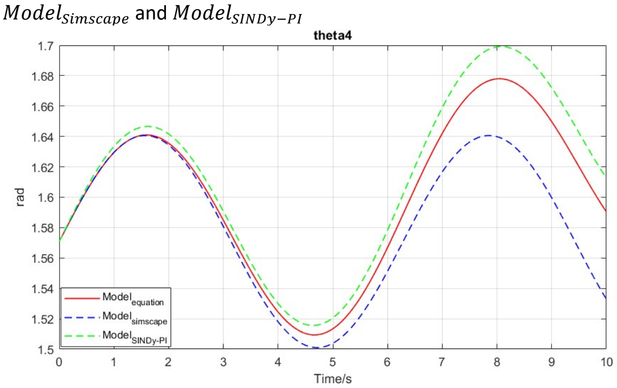


**Figure 6.** Comparison of joint angle $\theta_4$ among $Model_{equation}$, $Model_{Simscape}$ and $Model_{SINDy-PI}$

In this case, the SINDy-PI algorithm estimated equations using a candidate function library from $Model_{equation}$. In other words, the parameters are estimated so that resulting equations would fit $Model_{equation}$ for the best. Therefore, as can be seen in **Figures 3-6**, $Model_{SINDy-PI}$ shows a larger than $Model_{equation}$ from $Model_{Simscape}$. Future work could be adding terms representing damping elements to the SINDy-PI function library and training the algorithm with data directly from Simscape simulation.

## 5. Conclusion and future work

In this paper, a 2DoF simulated mechanism is developed in Matlab simscape with the purpose of applying SINDy-PI algorithm for parametric identification. Based on the results from open loop simulation, it is concluded that SINDy-PI has the ability to determine nonlinear system dynamics from simulated data to a good accuracy, resulting in RMSE values below 0.05.

The 2DoF mechanism was selected as a testing platform since it holds a good balance between non-linearity and complexity. Starting from the simplified equations and open-loop simulation, the use of SINDy-PI to determine system parameters will be further developed by: (1) development of a simscape multibody simulation of 2DoF mechanism along with closed loop controller, providing SINDy-PI with more a realistic data set to real world physics; (2) development of an experimental 2-DOF mechanism to use with SINDy-PI to evaluate accuracy of resulting equations; (3) finally we will test implementation of SINDy-PI directly on a Universal Robot UR5.

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