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# Capacitance analysis of a shielded sphere-flat capacitor in a high precision electrostatic force balance

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**Abstract**

The 2019 redefinition of the kilogram in the International System of Units, SI (Système International) through electrical metrology enables the direct realization of mass using precision balances. In an effort to extend the SI redefinition to new units of measure, next generation radionuclide metrology based on quantifying individual decay processes using superconducting transition edge sensors is developing. These methods require a cubic mm-scale liquid radionuclide solution sample. An electrostatic force balance can be used to measure the mass of this liquid, as required for a full traceability chain. This paper discusses the theory and experimental determination of the capacitance gradient of the electrodes of flat and spherical geometry used in an electrostatic force balance, including the treatment of the shield electrode and surface potentials.

## Introduction

Standard Reference Materials for radio activity, the Becquerel, are supplied as aqueous solutions of specific radionuclides characterized by massic activity in the units becquerel per gram of solution, Bq $kg^{-1}$ [1]. To realize a new method of defining radionuclide standards, the National Institute of Standards and Technology (NIST) put forth the True Becquerel project, creating a new paradigm for the realization and dissemination of radionuclide standards [2]. This project involves the development of an electrostatic force balance that measures absolute force, which is traceable to the SI, to determine the mass of a milligram-scale liquid radionuclide sample. The balance uses an electrostatic force generated by the capacitor electrodes, to compensate for the gravitational force of the radionuclide solution to maintain an equilibrium. The electrostatic force is calculated by the voltage applied to the electrodes and the capacitance gradient as a function of the gap between the electrodes. Since the goal of the balance is to measure mass with an uncertainty of less than 0.1 % for $k$ = 2, the uncertainty in the capacitance gradient must be smaller. The analysis of the chosen guarded sphere-flat capacitor geometry follows.

## 1. Capacitor design

In an electrostatic force balance, the gravitational force of the mass is compensated by an electrostatic force, generated by a capacitor. Knowing the characteristic of the capacitor and the applied voltage, the balance can be used as a primary force standard to measure absolute force without the need for calibration. The capacitor is arranged in a sphere-flat configuration see figure 1.

The sphere electrode is made from a non-magnetic 316 series stainless steel ball, while the flat electrode is a gold coated optical window with a flatness of λ/10 with $\lambda$ = 632.8 nm. The sphere electrode is fixed below the moving flat electrode in an electrically isolated configuration from surrounding parts. The flat electrode is connected to the guide mechanism of the electrostatic force balance and travelling along the x-axis, which changes the separation between the electrodes. To control the balance position during mass measurements, an interferometer beam is reflected from the top surface of the flat electrode. Due to the small manufacturing tolerances of the ball and optical window geometry, both the sphere and flat electrodes can be considered ideal. The use of a sphere-flat capacitor offers several advantages. The capacitance of the balance remains constant for transverse displacements in the z and y directions, even in the presence of thermal expansion or vibrations of the sphere or flat electrode. Additionally, the capacitance changes negligibly with angle, reducing the need for adjustments and minimizing noise and drift. As a result, only the angular alignment of the flat electrode to gravitational vector is relevant for mass measurement. To ensure accurate measurement of capacitance, a cylindrical guard surrounds the sphere, shielding the electrical field from other parts which are connected to the balance guide mechanism.

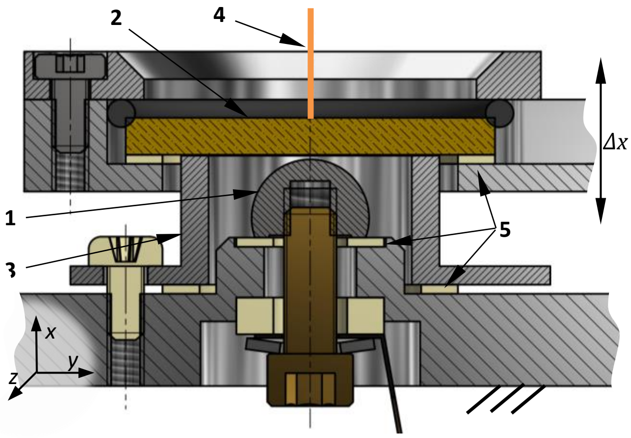


**Figure 1.** Cross sectional view of the design of sphere-flat capacitor which is integrated in an electrostatic force balance. 1 Sphere electrode, 2 Flat electrode, 3 Shield, 4 Interferometer Beam, 5 Isolation

## 2. Analytical description of the capacitor system

### *2.1. Three electrode system*

The electrostatic force of the capacitor is given by

$$F = -\frac{dW}{dx} \tag{1}$$

Where $W$ is the energy, stored in the capacitor and $x$ the displacement of the flat electrode. The energy of an enclosed electrostatic system is given by the following equation

$$W = \frac{1}{2}\boldsymbol{V}^T\boldsymbol{C}\boldsymbol{V} \tag{2}$$

where $V$ is a column vector of the sum of the applied and surface potentials, and $C$ is a matrix of self and mutual capacitance [5-6]. Since the capacitance is a function of $x$, the force can be written as

$$F = -\frac{1}{2}\boldsymbol{V}^T\frac{d\boldsymbol{C}}{dx}\boldsymbol{V} \tag{3}$$

To calculate the electrostatic force, one can formulate the voltage vector $V$ and capacitance matrix $C$ of the system. The capacitance can be described as an $N$ x $N$ matrix where $N$=3 is the number of electrodes. The capacitance matrix with elements $C_{ij}$, where $i,j \in = \{1,2,3\}$, is written as

$$\boldsymbol{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \tag{4}$$

and contains the self-capacitances for $i = j$ and the mutual capacitances for $i \neq j$.

Considering that:

- The matrix is symmetric: $C_{ij} = C_{ji}$
- The system is closed: $C_{ii} = \sum_{i \neq j} C_{ij}$
- Measured cross-capacitance is negative mutual capacitance: $C_{cij} = -C_{ij}$

[7] the matrix which describes the capacitance of the three-electrode capacitor can be written as

$$\boldsymbol{C} = \begin{pmatrix} C_{c12} + C_{c13} & -C_{c12} & -C_{c13} \\ -C_{c12} & C_{c12} + C_{c23} & -C_{c23} \\ -C_{c13} & -C_{c23} & C_{c13} + C_{c23} \end{pmatrix} \tag{5}$$

The potential vector $\boldsymbol{V}$ is a $N$ x 1 vector

$$\boldsymbol{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \tag{6}$$

To calculate the force, the capacitance matrix and potential vector in equation (3) can be substituted with equation (5) and (6). For $V_2 = V_3 = 0$, the force can be written as

$$F = -\frac{1}{2}V_1^2\left(\frac{dC_{c12}}{dx} + \frac{dC_{c13}}{dx}\right) \tag{7}$$

### *2.2. Surface potential in three electrode system*

The voltage measured in an electrostatic force balance includes both the voltage causing the force and a surface potential $\boldsymbol{V_s}$ from patch effect and surface contaminants. Therefore, the voltage vector of the system can be written as

$$\boldsymbol{V}_m = \boldsymbol{V} + \boldsymbol{V}_s = \begin{pmatrix} V_1 + V_{1s} \\ V_2 + V_{2s} \\ V_3 + V_{3s} \end{pmatrix} \tag{8}$$

For the measured force where $V_2 = V_3 = 0$,

$$\begin{aligned} F_m = &-\frac{1}{2}[(V_1 + V_{1s})^2 + V_{2s}(V_{2s} - 2V_1 - 2V_{1s})]\frac{dC_{c12}}{dx} \\ &-\frac{1}{2}[(V_1 + V_{2s})^2 + V_{3s}(V_{3s} - 2V_1 - 2V_{1s})]\frac{dC_{c13}}{dx} \\ &-\frac{1}{2}(V_{2s} - V_{3s})^2\frac{dC_{c23}}{dx} \end{aligned} \tag{9}$$

The difference in measured voltage $V_m$ and $V_1$, would result as a bias in the measured force $F_m$. This bias can be eliminated by reversing the polarity of the voltage, and a differential force measurement. The correction for surface potential follows a similar method developed for a two-electrode system [8], adapting it to a three-electrode capacitor. Since the balance is controlled at the same position when we apply a positive or negative potential, the force and therefore the absolute value of $V_1$ can be assumed as the same in both cases. The measured Voltage for a positive applied voltage $V_{Um+}$ and negative applied voltage $V_{Um-}$ for an unloaded case of the balance is therefore

$$V_{Um+} = \begin{pmatrix} V_{1U} + V_{1s} \\ V_{2U} + V_{2s} \\ V_{3U} + V_{3s} \end{pmatrix} \qquad V_{Um-} = \begin{pmatrix} -V_{1U} + V_{1s} \\ -V_{2U} + V_{2s} \\ -V_{3U} + V_{3s} \end{pmatrix} \tag{10}$$

Substituting equation (10) and (5) in equation (3) leads to

$$\begin{aligned} F_{Um+} = &-\frac{1}{2}[(V_{1U} + V_{1s})^2 + V_{2s}(V_{2s} - 2V_{1U} - 2V_{1s})]\frac{dC_{c12}}{dx} \\ &-\frac{1}{2}[(V_{1U} + V_{1s})^2 + V_{3s}(V_{3s} - 2V_{1U} - 2V_{1s})]\frac{dC_{c13}}{dx} \\ &-\frac{1}{2}(V_{2s} - V_{3s})^2\frac{dC_{c23}}{dx} \end{aligned} \tag{11}$$

$$\begin{aligned} F_{Um-} = &-\frac{1}{2}[(V_{1U} - V_{1s})^2 + V_{2s}(V_{2s} + 2V_{1U} - 2V_{1s})]\frac{dC_{c12}}{dx} \\ &-\frac{1}{2}[(V_{1U} - V_{1s})^2 + V_{3s}(V_{3s} + 2V_{1U} - 2V_{1s})]\frac{dC_{c13}}{dx} \\ &-\frac{1}{2}(V_{2s} - V_{3s})^2\frac{dC_{c23}}{dx} \end{aligned} \tag{12}$$

By averaging the positive and negative polarity, the cross terms $2V_{1U}V_{1s}$, $2V_{1U}V_{2s}$ and $2V_{1U}V_{3s}$ are eliminated, resulting in

$$\begin{aligned} F_{Uma} = &-\frac{1}{2}\left[{V_{1U}}^2 + (V_{1s} - V_{2s})^2\right]\frac{dC_{c12}}{dx} \\ &-\frac{1}{2}\left[{V_{2U}}^2 + (V_{1s} - V_{3s})^2\right]\frac{dC_{c13}}{dx} \\ &-\frac{1}{2}(V_{2s} - V_{3s})^2\frac{dC_{c23}}{dx} \end{aligned} \tag{13}$$

An analogous expression can be obtained for the balance's loaded state where a mass is placed on the balance

$$F_{Lma} = -\frac{1}{2}\left[{V_{1L}}^2 + (V_{1s} - V_{2s})^2\right]\frac{dC_{c12}}{dx} - \frac{1}{2}\left[{V_{2L}}^2 + (V_{1s} - V_{3s})^2\right]\frac{dC_{c13}}{dx} - \frac{1}{2}(V_{2s} - V_{3s})^2\frac{dC_{c23}}{dx} \quad (14)$$

By calculating the difference in force during unloaded and loaded cases, the surface potential terms $(V_{1s} - V_{2s})^2$, $(V_{1s} - V_{3s})^2$ and $(V_{2s} - V_{3s})^2$ cancels and the gravitational force of the mass can be written as

$$F_{mass} = F_{Lma} - F_{Uma} = -\frac{1}{2}\left(\frac{dC_{c12}}{dx} + \frac{dC_{c13}}{dx}\right)(V_{1L} - V_{1U})^2 \quad (15)$$

The above derivation shows that the bias caused by a constant surface potential is compensated, and the gravitational force of the mass is calculated as

$$F_{mass} = \frac{F_{Lm+} + F_{Lm-}}{2} - \frac{F_{Um+} + F_{Um-}}{2} \quad (16)$$

### *2.3. Reduction to two electrode system*

To avoid the measurements of two capacitance gradients $dC_{c12}/dx$ and $dC_{c13}/dx$, and the uncertainty in these measurements., it is preferred to reduce the number of electrodes in the system to two. By electrically connecting electrode two, the flat surface, and electrode three, the shield, the mass can be calculated as

$$F_{mass} = F_{Lma} - F_{Uma} = -\frac{1}{2}\left(\frac{dC_{c12}}{dx}\right)(V_{1L} - V_{1U})^2 \quad (17)$$

## 3. Fitting function for capacitance

From equation (17), any deviation in capacitance gradient has a proportional effect on the force. Since the goal of the electrostatic force balance is to measure mass with an uncertainty of less than 0.1 % for *k* = 2, the deviation between the capacitance gradient and the fitting function to estimate its value must be small.

The balance is moved in steps of 2 µm using a voice coil actuator and the capacitance is measured at each step. The position for each step can be controlled with an uncertainty of less than 25 nm while the capacitance is measured with an uncertainty of 15 aF. Assuming that a linear fit between these small increments is close to the true capacitance gradient allows us to compare the fitting functions. The grey dots in the first plot of figure 2 are the measured capacitances, while the second shows the linear capacitance gradient which is assumed as the true value. To reduce the drift effects, the capacitance is only measured at few positions during a regular measurement. For the current analysis, a subset of the measured data points is chosen for the input to the fitting functions, as shown by the black data points in figure 2. The figures show fitting functions in solid lines of different colours, their derivative, and the deviation from the capacitance gradient in percent. The chart illustrates that a polynomial fit does not describe the capacitance behaviour well enough. Therefore, Maxwell's equation describing a sphere-flat capacitor is used [9].

$$C = 4\pi\varepsilon r \sum_{n=1}^{\infty} \frac{\sinh\left(\ln\left(D + \sqrt{D^2 - 1}\right)\right)}{\sinh\left(n \ln\left(D + \sqrt{D^2 - 1}\right)\right)} \qquad D = \frac{r + x_g}{r} \quad (18)$$

Where $\varepsilon$ is the permeability of air, $r$ the radius of the sphere and $x_g$ the gap between the sphere and flat surface. Maxwell's equation does not describe the capacitance and its gradient completely since the sphere electrode is surrounded by a shield. This deviation can be minimized by adding a polynomial term to this equation, leading to a function resulting in the smallest fit residual.

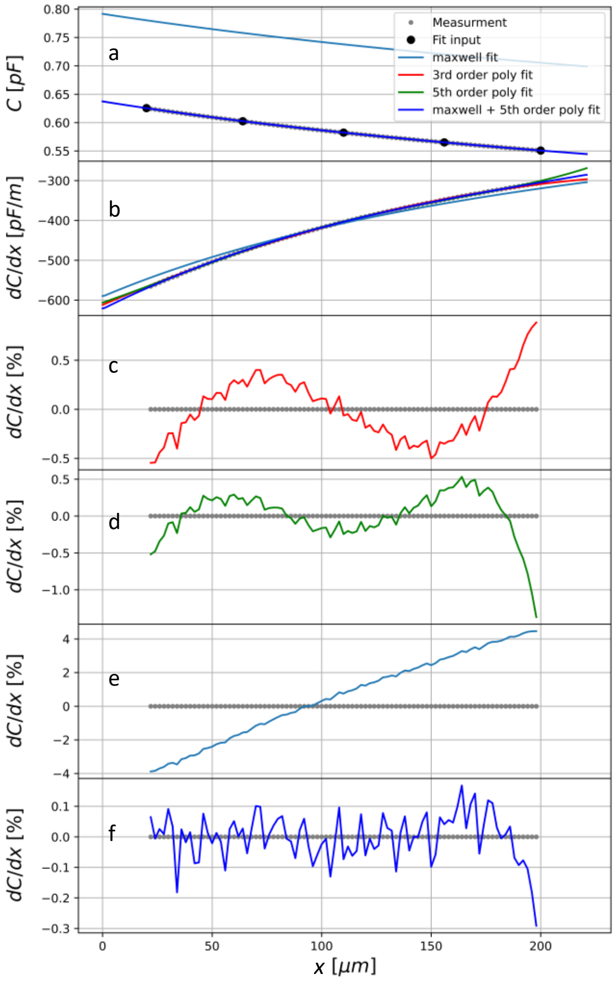


**Figure 2.** Comparison of the measured capacitance and the fitting function. (a) shows measured Capacitance, (b) shows the capacitance gradient calculated with various fitting functions, (c, d, e, and f) show fit residuals for the 3rd and 5th order polynomial, Maxwell, and combined Maxwell/5th order polynomial fits, respectively.

The noise in the relative residual is due to the noise during the capacitance measurement of 2 µm increments. The closeness of the fit depends on the range *x* and the number of positions the capacitance is measured. Linear or polynomial fits work better for a reduced range of *x* and increasing the number of positions where capacitance is measured. However, the uncertainty in capacitance gradient would increase for a smaller range. Also, the number of positions where the capacitance is measured and the time for each measurement should be as small as possible to minimize drift.

## 4. Drift in capacitance

To estimate the effect of drift in the capacitance gradient measurement, a measurement was done over a time span of 60 hours under stable laboratory conditions while the position of the balance was controlled to a setpoint using the voice coil actuator.

Each datapoint in the timeseries data of figure 3 is a simultaneous measurement of position, capacitance or temperature averaged over 3 minutes. Since the standard deviation in capacitance over 3 minutes is about 15 aF, the drift is not significant for the current measurements. Therefore, the capacitance measurement time can be increased.

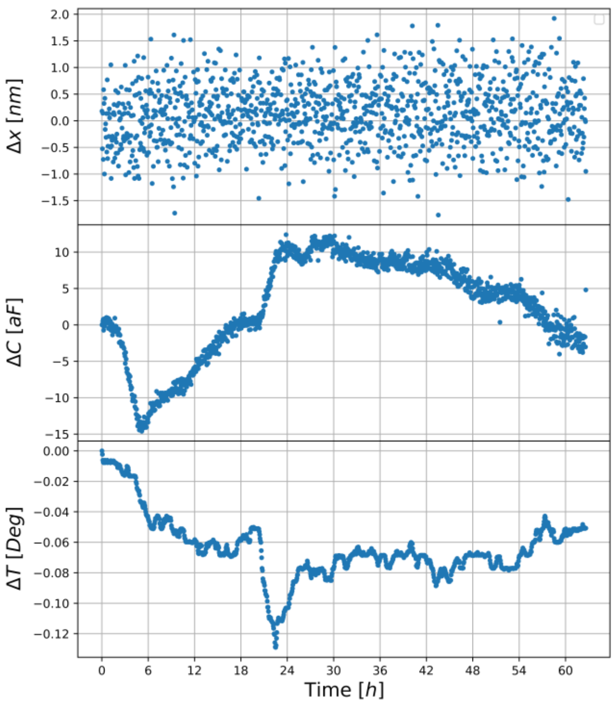


**Figure 3.** Drift of capacitance over time

The optimum found by an Allan deviation plot is about 15 minutes, see Figure 4.

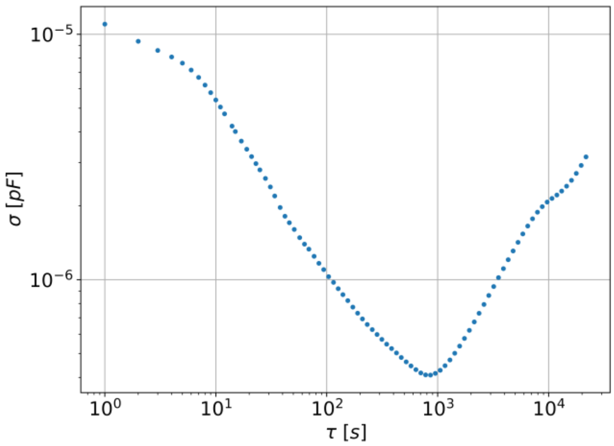


**Figure 4.** Allan variance of capacitance data

### 5. Conclusion

This paper discusses the capacitance analysis of a shielded sphere-flat capacitor used in an electrostatic force balance for mass metrology. The goal of this project is to develop a method for realizing and disseminating radionuclide standards. The electrostatic force balance is designed to measure the mass of a milligram-scale liquid radionuclide sample with a relative uncertainty of less than 0.1 %. The paper describes the design of the capacitor, the analytical description of the systems capacitance and the experimental determination of the capacitance and its gradient, using fitting functions. The advantages of using a sphere-flat capacitor in the balance are highlighted, including its constant capacitance for transverse displacements, negligible capacitance changes with angle, and reduced drift. Using a third electrode to shield and concentrate the electrostatic field which improves the capacitance measurement, but still allows for surface potential compensation using the reversed polarity method. To find the optimum number of measurements and measurement time for the capacitance gradient measurement, further investigations can be performed.

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