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# Mobility analysis of folded sheet flexure based 2-DOFs rotational platform

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## Abstract

A folded sheet flexure provides a translational constraint along the fold line. In this paper, we designed a floating platform using four folded sheet flexures to achieve two rotational DOFs. Constraint line approach is used to design the flexure motion stage. The mobility of the platform is analysed using screw algebra. A wrench is defined along the constraint line between the floating platform and the base. Various configurations using four folded sheet flexures are considered, and subsequently constrained space matrix is obtained. Mobility of the platform is thus calculated using the reciprocity condition of the screws. A candidate design is obtained for two rotational DOFs. A prototype is fabricated using the rapid prototyping technique. Constraints and DOFs are verified using different methods and compared. Range of motion is compared along different directions. The developed methodology will be further used to design a 2-DOFs floating platform for various tracking applications. Screw algebra can be used for the mobility analysis of any general configuration using different numbers of folded sheet flexures and an optimised solution may be obtained for a given application.

2-DOF floating platform, folded sheet flexure, screw algebra, mobility analysis

#### 1. Introduction

To locate a point in a spherical coordinate system, three quantities are needed: the radial distance, the azimuthal angle of the point's orthogonal projection, and the polar angles from a fixed zenith direction [1]. The systems with two degrees of freedom can effectively follow a point in space [2-5].

Rolling and sliding contacts are used to achieve mobility in rigid body mechanisms, while the elastic deformation of thin segments is used for motion in the complaint mechanisms [4-6]. Conventionally, a universal joint is used to achieve two degrees of freedom (DOF) in mechanical systems.

In this paper, a floating platform using four folded sheet flexures is designed to achieve two rotational degrees of freedom about its orthogonal axes. Mobility analysis is performed using the screw algebra for various configurations of folded sheet flexure (FSF) in Sec. 2. Finite Element Analysis (FEA) software (ANSYS) is used for modal analysis to verify the results and discussed in Sec. 3, and the conclusion and future work are written in Sec. 4 and Sec. 5 respectively.

## 2. Freedom of a rigid body

A rigid body in space has six degrees of freedom. Hence it can rotate and translate about its three orthogonal axes. Physical contact between the rigid body applies the constraint. Thus, it reduces the DOF. The direction of the given constraint is referred to as the constraint line.

In screw theory, a constraint is expressed by a wrench, while a twist expresses a DOF,  $\hat{T} = (\Omega | V)$  and  $\hat{W} = (F | M)$  respectively [8]. Hence a freedom space of a rigid body in a space having six DOF can be represented by  $\Pi_T = [\hat{T}_1, \hat{T}_2, \hat{T}_3, \hat{T}_4, \hat{T}_5, \hat{T}_6]^T$ .

## 2.1. Mobility analysis of a folded sheet flexure

A folded sheet flexure constrains only one translation along its fold axis, which is similar to a wire flexure [2-3]. An FSF is shown in Fig. 1.



Figure 1. A folded sheet flexure (FSF) with DOFs and one constraint along the fold line which is shown with a dashed line

In Fig. 1, one end of the FSF is fixed, while the other end is set to be a free end. The origin of an orthogonal coordinate system O' is defined at the centre of the FSF and x – axis is collinear with the fold axis of the FSF. Line segment  $C_1$  represents the constraint direction [6].

Wrench along  $C_1$  for the FSF is written as

 $\widehat{W}_1 = (1 \ 0 \ 0 \mid 0 \ 0 \ 0)$ 

The general motion of a rigid body is represented by a general twist  $\hat{T}$  as

(1)

$$\widehat{T} = \left( \Omega_x \, \Omega_y \, \Omega_z \right| V_x \, V_y \, V_z) \tag{2}$$

When a rigid body is constrained, the instantaneous motion is possible in a tangent plane normal to the constraint. Hence, the reciprocity condition can be used to write the screws [8-9] as  $\widehat{T} \circ \widehat{W} = F \cdot V + M \cdot \Omega = 0$  (3)

On Applying the reciprocity condition, we get  $V_x = 0$ , and therefore the general twist for the given case is:

(4)

 $\widehat{T} = \left( \Omega_x \, \Omega_y \, \Omega_z \right| \, 0 \, V_y \, V_z)$ It can be further written as five independent twists as  $\widehat{T}_1 = (0 \ 0 \ 0 \ | \ 0 \ 0 \ 1)$  $\widehat{T_2} = (0 \ 0 \ 0 \ | \ 0 \ 1 \ 0)$  $\widehat{T_3} = (1 \ 0 \ 0 \ | \ 0 \ 0 \ 0)$  $\hat{T}_4 = (0\ 1\ 0\ |\ 0\ 0\ 0)$  $\widehat{T_5} = (0\ 0\ 1\ |\ 0\ 0\ 0)$ 

Hence the freedom space Matrix of the FSF is written a  $\Pi_{\mathrm{T}} = [\widehat{T}_{1}, \widehat{\mathrm{T}}_{2}, \widehat{\mathrm{T}}_{3}, \widehat{T}_{4}, \widehat{\mathrm{T}}_{5}]^{T}$ 

The above mobility analysis shows that only one translational motion is constrained, but the body can still translate in the y & z - direction and rotate about all three orthogonal axes. The same can be obtained through inspection as well.

## 2.2. Mobility analysis of folded sheet flexure-based 2-DOF rotational platform



Figure 2. A physical arrangement of screws in a suspended platform where all constraints are oriented randomly

A floating stage is designed with a combination of four FSFs, one end of the FSF is fixed with the base, while the other end is attached to the suspended platform. A global cartesian coordinate system is defined at point O. Constraint line 'S<sub>i</sub>' where i = [1,4] is defined along the constraint direction of each FSF. The point ' $a_i$ ' is located at the base where a local cartesian coordinate system is defined. The angle between line ' $Oa_i$ ' and the y-axis is represented by ' $\alpha_i$ '. ' $S_z$ ' is the projection of the vector  $S_i$  on xy plane and ' $\theta_i$ ' and ' $\emptyset_i$ ' is the azimuthal angle of the point's orthogonal projection and the polar angles from a fixed (z – axis) zenith direction, respectively. ' $r_i$ ' is the distance of point ' $a_i$ ' from the origin 'O', as shown in Fig. 2.

Wrenches for the general case are written as  

$$\widehat{W}_{i} = (F_{xi} - F_{yi} - F_{zi})M_{xi} - M_{yi} - M_{zi})$$

$$\widehat{W}_{l} = (F_{xj} \quad F_{yj} \quad F_{zj} | M_{xj} \quad M_{yj} \quad M_{zj})$$
(5)  

$$F_{ij} \text{ is the force along } i^{th} - \text{axis due to } j^{th} \text{ constraint.}$$
where  $i = (x \ y \ z) \text{ and } j = (1 \ 2 \dots \ n)$ 

The constraint space matrix is written as

 $\prod_{W} = \left[ \widehat{W_{1}}, \widehat{W_{2}}, \widehat{W_{3}}, \widehat{W_{4}} \right]^{T}$ 

Now, the reciprocity condition is used to find the twists  $\hat{T}$ . For the given case, The Freedom space matrix is written as

(6)

 $\Pi_{\mathrm{T}} = [\widehat{T}_1, \widehat{\mathrm{T}_2}, \dots, \widehat{\mathrm{T}_f}]^{\mathrm{T}}$ 

Where f is the order of DOF.

Further, the Adjoint transformation of screws is used to find the screws at specific locations [9].

## 2.2.1. Case 1: all constraints are parallel

In this case, all four FSFs are parallels to the z-axis. Hence, for the given configuration, i = [1,4] the angles  $\mathbf{\Theta}_i = \mathbf{\Phi}_i = \mathbf{0}^o$  and  $\boldsymbol{\alpha}_i = \mathbf{60}^o$ . The screw is taken as unit length and  $r_i = 50~mm$ , as shown in Fig. 3.



Figure 3. A physical arrangement of screws in a suspended platform where all constraints are parallel (Case 1)

Wrenches for case 1 with respect to the global coordinate system are given as

$$\begin{split} \widehat{W}_{1} &= (0 \quad 0 \quad 1 \mid 25 \quad 43.3 \quad 0) \\ \widehat{W}_{2} &= (0 \quad 0 \quad 1 \mid -25 \quad 43.3 \quad 0) \\ \widehat{W}_{3} &= (0 \quad 0 \quad 1 \mid -25 \quad -43.3 \quad 0) \\ \widehat{W}_{4} &= (0 \quad 0 \quad 1 \mid 25 \quad -43.3 \quad 0) \\ \text{The constraint space matrix is written as} \\ \prod_{w} &= \left[\widehat{W_{1}}, \widehat{W_{2}}, \widehat{W_{3}}, \widehat{W_{4}}\right]^{T} \\ \prod_{w} \text{ has rank 3. It's not a full rank matrix, and therefore, it is an} \end{split}$$

over-constrained design. Hence, for a general twist  $\hat{T}$ , Eqn. 3 is satisfied when  $V_Z = \Omega_X = \Omega_y = 0$ 

Therefore, the general twist for the given case is 

$$I = (0 \ 0 \ \Omega_Z | \ V_X \ V_Y \ 0)$$

It can be further written as three independent twists as

 $\widehat{T}_1 = (0\ 0\ 0\ |\ 1\ 0\ 0)$ 

$$l_2 = (0\ 0\ 0\ |\ 0\ 1\ 0)$$

 $\widehat{T_3} = (0\ 0\ 1\ |\ 0\ 0\ 0)$ 

From Eqn. 6, the Freedom space matrix is written as 31<sup>T</sup>

$$\Pi_{\mathrm{T}} = [\widehat{T}_1, \widehat{\mathrm{T}}_2, \widehat{\mathrm{T}}_3]$$

It shows that the arrangement can rotate about a single axis while it can translate in two orthogonal axes, as shown in Fig.3.

#### 2.2.2. Case 2: all constraints are concurrent



Figure 4. A physical arrangement of screws in a suspended platform where all constraints are concurrent (Case 2)

In this case, all four FSFs are oriented in such a way that constraint directions are concurrent at a point on z —axis, as shown in Fig. 4. For the given configuration, we have chosen the angles  $\theta_i = 30^o$ ,  $\varphi_i = 49.11^o$  and  $\alpha_i = 60^o$ . The screw is taken as unit length and  $r_i = 50 mm$ , where i = [1,4].

Wrenches for case 2 with respect to the global coordinate system are given as

 $\widehat{W}_1 = (0.65 - 0.38 \ 0.65 \mid 16.4 \ 28.3 \ 0)$  $\widehat{W}_2 = (0.65 \quad 0.38 \quad 0.65 \mid -16.4 \quad 28.3 \quad 0)$ (9)  $\widehat{W}_3 = (-0.65 \quad 0.38 \quad 0.65 \mid -16.4 \quad -28.3 \quad 0)$  $\widehat{W}_4 = (-0.65 \quad -0.38 \quad 0.65 \mid 16.4 \quad -28.3 \quad 0)$ The constraint space matrix is written as

 $\prod_{W} = \left[\widehat{W_{1}}, \widehat{W_{2}}, \widehat{W_{3}}, \widehat{W_{4}}\right]^{T}$ 

 $\prod_{w}$  has rank 3. This configuration is also an over-constrained design.

Hence, for a general twist  $\hat{T}$ , Eqn. 3 is satisfied when  $V_Z = 0, V_X = -43.5\Omega_Y$  and  $V_Y = 43.16\Omega_X$ 

Therefore, the general twist for the given case is

 $\widehat{T} = (\Omega_X \,\Omega_Y \,\Omega_Z \mid -43.5\Omega_y \,43.16\Omega_x \,0)$ 

It can be further written as three independent twists as

 $\widehat{T}_1 = (0\ 1\ 0\ |\ -43.5\ 0\ 0)$ 

 $\widehat{T_2} = (1 \ 0 \ 0 \ | \ 0 \ 43.16 \ 0)$   $\widehat{T_2} = (0 \ 0 \ 1 \ | \ 0 \ 0 \ 0)$ 

It shows that the arrangement can freely rotate about Z - axis, but the translation along X - axis is coupled with rotation about Y - axis, and translation along Y - axis is coupled with rotation in about X - axis and vice versa.

At the point of concurrence, wrenches for case 2 are given as

 $\hat{W}_1 = (0.65 - 0.38 \ 0.65 | 0 \ 0 \ 0)$  $\widehat{W}_2 = (0.65 \quad 0.38 \quad 0.65 \mid 0 \quad 0 \quad 0)$ (10) $\widehat{W}_3 = (-0.65 \quad 0.38 \quad 0.65 \mid 0 \quad 0 \quad 0)$  $\widehat{W}_4 = (-0.65 \ -0.38 \ 0.65 \ | \ 0 \ 0 \ 0)$ At the point of concurrence, for a general twist  $\hat{T}$ , Eqn. 3 is satisfied when  $V_X = V_Y = V_Z = 0$ 

Therefore, the general twist for the given case is

 $\hat{T} = \left( \Omega_x \, \Omega_y \, \Omega_z \right| \, 0 \, 0 \, 0)$ 

It can be further written as three independent twists as

 $\widehat{T_{R1}} = (1\ 0\ 0\ |\ 0\ 0\ 0)$ 

 $\widehat{T_{R2}} = (0\ 1\ 0\ |\ 0\ 0\ 0)$ 

 $\widehat{T_{RZ}} = (0\ 0\ 1\ |\ 0\ 0\ 0)$ 

From Eqn. 6, the freedom space matrix is written as

 $\Pi_{\mathrm{T}} = [\widehat{T_{\mathrm{R1}}}, \widehat{T_{\mathrm{R2}}}, \widehat{T_{\mathrm{R2}}}]^{\mathrm{T}}$ 

It shows that the arrangement can freely rotate about all orthogonal axes at the point of concurrence, as shown in Fig. 4, i.e., it acts as a spherical joint.

## 2.2.3. Case 3: constraints intersect at two different points



Figure 5. A physical arrangement of screws in a suspended platform where only two screws intersect at a point (Case 3)

In this case, all four FSFs are oriented in such a way that  $S_1 \& S_2$  intersects at a point while  $S_2 \& S_3$  intersects at a different point on XZ —plane, as shown in Fig. 5. For the given configuration, the angles  $\theta_i = 49.11^o$ ,  $\phi_i = 41.41^o$  and  $\alpha_i =$  $60^{\circ}$ . The screw is taken as unit length and  $r_i = 50$  mm, where i = [1,4].

Wrenches for case 3 with respect to the global coordinate system are written as

 $\widehat{W}_1 = (0.43 - 0.5 0.75 | 18.7 32.5 10.8)$  $\widehat{W}_2 = (0.43 \quad 0.5 \quad 0.75 \mid -18.7 \quad 32.5$ -10.8) (11) $\widehat{W}_3 = (-0.43 \quad 0.5 \quad 0.75 \mid -18.7 \quad -32.5 \quad 10.8)$  $\widehat{W}_4 = (-0.43 \quad -0.5 \quad 0.75 \mid 18.7 \quad -32.5 \quad -10.8)$ The constraint space matrix is written as  $\prod_{W} = \left[\widehat{W_{1}}, \widehat{W_{2}}, \widehat{W_{3}}, \widehat{W_{4}}\right]^{T}$  $\prod_{w}$  has rank 4. The matrix is full rank and therefore, the design is exactly constrained. For a general twist  $\hat{T}$ , Eqn. 3 is satisfied when  $V_Z = \Omega_Z = 0$ ,  $V_X = -75.58\Omega_Y$  and  $V_Y = 37.4\Omega_X$ 

Therefore, the general twist for the given case is

 $\hat{T} = \left(\Omega_x \,\Omega_y \,0\right| -75.58\Omega_y \,37.4\Omega_x \,0)$ 

It can be further written as two independent twists as  $\widehat{T}_1 = (0\ 1\ 0\ |\ -75.58\ 0\ 0)$ 

 $\widehat{T_2} = (1\ 0\ 0\ |\ 0\ 37.4\ 0)$ 

From Eqn. 6, the freedom space matrix is written as:  $\Pi_{\mathrm{T}} = [\widehat{T}_1, \widehat{\mathrm{T}}_2]^{\mathrm{T}}$ 

It shows that the translation along X - axis is coupled with the rotation about Y - axis, and translation along Y - axis is coupled with rotation in about X - axis and vice versa, as shown in Fig. 5.

## 3. Results and discussion

Analytical results show that in Case 1, the floating platform translates along X and Y –axes and rotates about Z –axis. The translation about Z —axis, and the rotation about X and Y -axes are constrained. In Case 2, all translations are constrained while the rotations are allowed about its orthogonal axes at the concurrent point of constraint lines. Case 2 is identical to a spherical joint. Case 3 has two rotational DOFs about  $R_1 \& R_2 - axes$ . Freedom, general twist, and wrenches of the FSF-based floating platform for different cases are summarized in Tab. 1.

Cases	Freedom Symbol	Twist $[\widehat{T}]$	Wrench $[\widehat{W}]$	
Case 1	2P-R	$[\widehat{P_X}  \widehat{P_Y}  \widehat{R_Z}]$	$[\widehat{P_Z} \ \widehat{R_X} \ \widehat{R_Y}]$	
Case 2	3 <i>R</i>	$[\widehat{R_1}  \widehat{R_2}  \widehat{R_Z}]$	$[\widehat{P_X}  \widehat{P_Y}  \widehat{P_Z}]$	
Case 3	2R	$[\widehat{R_1}\widehat{R_2}]$	$[\widehat{P_X} \ \widehat{P_Y} \ \widehat{P_Z} \ \widehat{R_Z}]$	

Table 1 Motion and constraint space of FSF-based floating platform

Where  $R_i \& P_i$  represent the rotation and translation about *i<sup>th</sup>* –axis, respectively.

Modal analysis was performed on ANSYS Workbench to obtain the frequencies of different vibration modes. The material of the FSF is structural steel with Young's Modulus of 210 GPa and Poisson's ratio of 0.3. The modal frequencies of different modes are summarized in Tab. 2

The stiffness of the system for a particular mode,  $k_e$  is written as

$$k_e = m_e \omega_e^2 \tag{13}$$

Where  $\omega_e$  and  $m_e$  is the modal frequency and mass for a particular mode respectively. The stiffnesses are compared using Eqn. 13. Mobility of the platform for different cases can be obtained and verified by comparing the ratio of stiffnesses.



Figure 6. Representation of the Mode shape analysis of (a)case 1, (b) case 2, and (c) case 3

Table	2.	Modal	frequency	of	different	modes	of	FSF-based	floating
platfo	rm								

Modal frequency mode	Magnitude (Hz)			
	Case 1	Case 2	Case 3	
Translation motion in x-axis	53.68	291.33	500.20	
Translation motion in y-axis	53.86	173.58	243.62	
Translation motion in z-axis	535.13	279.52	346.8	
Rotation motion about x-axis	617.03	70.99	64.48	
Rotation motion about y-axis	1199	74.22	77.80	
Rotation motion about z-axis	98.62	89.48	184.22	

From Tab. 2, it is observed that for case 1, the configuration is compliant in translation along X, Y —axes and rotation about Z —axis. Similarly, for case 2, the configuration has less stiffness for three rotations at the point of concurrency, while the stiffness is high for all the translations. Case 3 is stiffer for all translations and rotation about Z —axes, but it has less stiffness for the rotation about X, Y —axes.

Both analytical and modal analysis results are summarized in Tab. 3. It has been observed that DOFs of all the cases are the same in both analytical and modal analysis results. These methods can be further used to design a mechanism with two remote axes of rotations for the different tracking applications, as shown in case 3. Space tracking, radar positioning, astronomical telescope mounts, laser scanning and positioning, etc., are the primary area of application.

 Table 3 Comparison of Analytical and modal analysis results of different cases of FSF-based floating platform.

FSF-based floating	Freedom space matrix			
platform	Analytical	Modal analysis		
Case 1	$[\widehat{P_X} \ \widehat{P_Y} \ \widehat{R_Z}]$	$[\widehat{P_X} \ \widehat{P_Y} \ \widehat{R_Z}]$		
Case 2	$[\widehat{R}_1  \widehat{R}_2  \widehat{R}_Z]$	$[\widehat{R}_X \ \widehat{R}_Y \ \widehat{R}_Z]$		
Case 3	$[\widehat{R_1} \ \widehat{R_2}]$	$[\widehat{R_X} \ \widehat{R_Y}]$		

## 4. Conclusion

We used screw theory to perform a mobility pattern analysis for three different configurations of FSF-based floating platforms in which wrenches are defined along the constraint directions of FSF. The constraint space matrix is not a full rank in the case of over-constrained configurations. Further, reciprocity condition of the screw is used to find the twists for individual cases and adjoint transformation of the screw was used to get the screws at a desired location. Modal analysis was performed to validate the analytical results. Analytical and modal analysis results were compared and found to be similar. Screw based mobility analysis can be further used to design suitable flexurebased mechanism for space tracking applications.

## 5. Future work

A physical FSF-based floating platform can be developed to study the effect of FSF orientation on the floating platform mobility. Further optimisation will be performed to find the orientation of FSF for a specific application.

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