euspen's 18th International Conference & Exhibition, Venice, IT, 2018
www.euspen.eu

# A Hybrid Trust Region Algorithm for Minimum Zone Fitting of FreeForm Surfaces

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**Abstract**

Optical elements with non-symmetric features, also known as freeform optical shapes are very popular in the domain of optical design. Unlike standard optical surfaces such as spherical or aspherical lenses, freeform shapes provide superior optical performances especially for aberration control at multiple locations. However, a hurdle to the development of these elements is the efficient metrological characterization. Because of their complicated description, freeforms characterization is challenging either during the process of measurement or data fitting. Data fitting complication increases when adopting infinite norm ($L_\infty$) instead of least squares ($L_2$) to determine the minimum zone (MZ). In this paper, a new algorithm called hybrid trust region (HTR) for data fitting of freeforms according to MZ is presented. Validation of the proposed algorithm was performed using reference data. The newly introduced algorithm proved to be more efficient than those reported in existing methods.

Dimensional metrology, Freeform, Minimum Zone Fitting, $L_\infty$-norm, Optimization

## 1. Introduction

Freeform optics are very important elements in optical design since they provide superior performances compared to spherical and aspherical lenses (fig. 1). Emergence of freeform has grown up thanks to advances made in manufacturing techniques. However, a hurdle to the development of this class of elements is the efficient metrological characterization. The complex shape of freeform makes the process of measurement and data fitting more challenging. The choice of $L_\infty$-norm fitting (often called the Chebyshev or minimax norm) instead of $L_2$ (least squares) makes the fitting more complex.

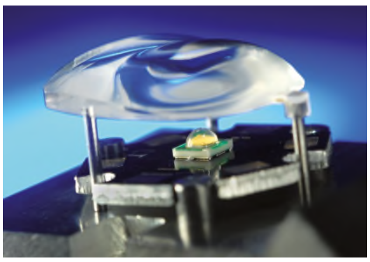

**Figure 1.** LED with injection-molded freeform optic [1]

The aim of data fitting is to infer information about conformance of the measured part to specifications. One criterion of conformance is the minimum zone (MZ). To determine the minimum zone, form deviations, which are the orthogonal distances between measured points and a reference surface, must be determined. Peak-to-valley (PV) is defined as the difference between the maximum and minimum form deviations. Minimum zone is the least value of PV among all choices of reference surfaces. It is to be noted that Chebyshev fitting gives the exact minimum zone since it directly minimizes the PV.

Minimum zone determination was extensively studied for simple geometries, *viz.* lines, planes, circles, cylinders, spheres, etc. and a number of techniques were used (computational geometry techniques, simplex methods, linear/nonlinear programming…). Few works were achieved for complex geometries such as aspherical lenses [2, 3]. Many approaches like smoothing techniques, primal-dual interior point methods, genetic algorithms, etc. could be used to determine the minimum zone.

In this work, a hybrid trust region algorithm is used to determine the minimum zone for freeform. In section 2, an overview of the introduced algorithm is presented. Validation of the proposed algorithm was performed on simulated (reference) data in section 3.

## 2. Hybrid Trust Region algorithm (HTR)

The problem of minimum zone fitting relies on the infinite norm. Here, the absolute value of maximum form deviation over all data points is minimized with respect to all choices of reference surfaces.

Suppose given are a set of $m$ measured data points $\{\boldsymbol{P}_i\}_{1\le i\le m}$ and their corresponding orthogonal projections $\{\boldsymbol{Q}_i\}_{1\le i\le m}$ onto a surface described by the mean of an implicit equation $f(\boldsymbol{q},\boldsymbol{s})=0$, with $\boldsymbol{q}=(x,y,z)$ are the coordinates of a given point on the surface and $\boldsymbol{s}$ are the surface' shape parameters. The minimum zone fitting problem could be formulated as in eq. (1).

$$\min_{x} \phi(\boldsymbol{x}) \text{ where } \phi(\boldsymbol{x}) = \max_{1\le i\le m} f_i(\boldsymbol{x}) \quad (1)$$

$f_i$ denotes the Euclidean distance between the point $P_i$ and its corresponding orthogonal projection $Q_i$, $x \in R^n$ could be either the set of intrinsic shape parameters $\boldsymbol{s}$, or the motion parameters $\boldsymbol{m}$: rotation and translation applied to $\{\boldsymbol{P}_{\mathrm{i}}\}$.

The main idea of the hybrid trust region algorithm consists of performing either trust region step, line search step or curve search step according to the specific situation faced at each iteration. This enables to avoid solving the trust region problem many times. For each iteration, a first point consists of obtaining a trust region trial step $\boldsymbol{d}_{\mathrm{k}}$ by solving the quadratic problem $\mathrm{QP}(\boldsymbol{x}_{\mathrm{k}}, \mathrm{B}_k)$ given in (2).

$$\begin{aligned} &\min_{(\boldsymbol{d},z)\in R^{n+1}} \frac{1}{2} < \boldsymbol{d}, \boldsymbol{B}_k \boldsymbol{d} > + z = M_k(\boldsymbol{d}, z), \\ &s.t. < \nabla f_i(\boldsymbol{x}_k), \boldsymbol{d} > -z \leq \phi(\boldsymbol{x}_k) - f_i(\boldsymbol{x}_k), \quad i = 1, \dots, m \\ &\|\boldsymbol{d}\|_\infty \leq \Delta_k \end{aligned} \tag{2}$$

where $\boldsymbol{B}_k$ is $n$ by $n$ symmetric positive definite matrix, $\Delta_k$ is the parameter defining the trust region domain, $z$ is an introduced parameter depending on the first derivative of the objective function $\phi$, $\nabla f_i$ is the gradient of the function $f_i$ and $<.,.>$ defines the dot product.

The trust region domain is defined using $L_\infty$ instead of $L_2$ so as $(QP)$ becomes an easily-solved quadratic problem. It should be mentioned that the proposed $(QP)$ in (2) is always feasible since $(0,0)$ is a feasible solution. This problem could be solved using classical methods adapted to quadratic problems such as interior point method [4].

If the resulting trust region trial step $\boldsymbol{d}_k$ could not be accepted, a corrected step $\boldsymbol{d}_k + \widetilde{\boldsymbol{d}}_k$ is determined by solving the problem $\widetilde{\mathrm{QP}}(\boldsymbol{x}_k, B_k)$ given in (3).

$$\begin{aligned} &\min_{(\tilde{\boldsymbol{d}},z)\in R^{n+1}} \frac{1}{2} < \boldsymbol{d}_k + \tilde{\boldsymbol{d}}, \boldsymbol{B}_k(\boldsymbol{d}_k + \tilde{d}) > + \tilde{z} = \widetilde{M}_k(\tilde{\boldsymbol{d}}, \tilde{z}), \\ &s.t. < \nabla f_i(\boldsymbol{x}_k), \tilde{\boldsymbol{d}} > -\tilde{z} \leq \phi(\boldsymbol{x}_k + \boldsymbol{d}_k) - f_i(\boldsymbol{x}_k + \boldsymbol{d}_k), i = 1, \dots, m \\ &\|\boldsymbol{d}_k + \tilde{\boldsymbol{d}}\|_\infty \leq \Delta_k \end{aligned} \tag{3}$$

If neither the initial trust region step $\boldsymbol{d}_k$ nor the corrected step $\boldsymbol{d}_k + \widetilde{\boldsymbol{d}}$ could be acceptable in trust region scheme, a line search along $\boldsymbol{d}_k$ or a curve search is performed if the $\boldsymbol{d}_k$ is a descent direction. If it is not the case, a curve search is used to find a step length $t_k$ that verifies (4) where $\alpha \in (0, 1/2)$.

$$\phi(\boldsymbol{x}_k + t_k \boldsymbol{d}_k + t_k^2 \widetilde{\boldsymbol{d}}_k) \leq \phi(\boldsymbol{x}_k) - \alpha t_k < \boldsymbol{d}_k, B_k \boldsymbol{d}_k > \tag{4}$$

where $\boldsymbol{d}_k$ is the solution of (2) and $\widetilde{\boldsymbol{d}}_k$ is the solution of (3). In the case $\|\boldsymbol{d}_k\| \leq \|\widetilde{\boldsymbol{d}}_k\|$, $\widetilde{\boldsymbol{d}}_k$ should be taken to be 0.

For $B_k$ update, the Powell's modification of BFGS formulas is used [5]. A detailed implementation of the algorithm is given in [6].

## 3. Algorithm validation

Validation of the proposed algorithm was performed using reference data generated according to a method developed by Forbes in [7]. The main idea of the said method is based on the formulation of sufficient optimality conditions for problem (1) and then inferring corresponding data.

The test was performed on a freeform that has no degree on invariance described by the explicit equation given in (5)

$$\begin{aligned} z = \alpha_1(x^3 + y^3) + \alpha_2(xy^2 + x^2y) + \alpha_3(xy^4 + x^4y) \\ + \alpha_4(x^2y^3 + x^3y^2) \end{aligned} \tag{5}$$

A number of data points with known minimum zone value are generated and submitted to the algorithm. In order to assess the result, returned value given by the algorithm under test is compared to the known minimum zone value. In this test, only motion parameters are sought, shape parameters values are supposed given. Chosen $\alpha_i$ coefficients for each data set, minimum zone value and the algorithm result are given in table 1.

**Table 1** $\boldsymbol{\alpha}_{\mathrm{i}}$ coefficients for each data set, minimum zone value and the algorithm result

| Data set | Shape parameters | Minimum zone value | Algorithm result |
|---|---|---|---|
| 1 | $\alpha_1 = 4.1\ 10^{-4}$<br>$\alpha_2 = -6.8\ 10^{-4}$<br>$\alpha_3 = -2.4\ 10^{-4}$<br>$\alpha_4 = 9.2\ 10^{-4}$ | $5\ 10^{-3}$ | $5.00002\ 10^{-3}$ |
| 2 | $\alpha_1 = 1.5\ 10^{-4}$<br>$\alpha_2 = 8.0\ 10^{-4}$<br>$\alpha_3 = -4.2\ 10^{-4}$<br>$\alpha_4 = 6.5\ 10^{-4}$ | $10^{-3}$ | $1.00087\ 10^{-3}$ |
| 3 | $\alpha_1 = -5.6\ 10^{-4}$<br>$\alpha_2 = 2.1\ 10^{-4}$<br>$\alpha_3 = 1.6\ 10^{-4}$<br>$\alpha_4 = 3.2\ 10^{-4}$ | $5\ 10^{-4}$ | $5.00037\ 10^{-4}$ |

Minimum zone results obtained by the hybrid trust region algorithm match reference values. In fact, the difference between reference values and HTR results is of the fourth order. This could be justified by the use of iterative methods to calculate orthogonal distances between measured data and the reference surface. Nevertheless, this difference could be acceptable. The algorithm proves to work well for initial alignments up to 15° in all directions ($x$, $y$ and $z$).

## 4. Conclusion

In this paper a new algorithm for minimum zone data fitting of freeform was presented. The proposed method makes use of a hybrid algorithm based on a combination of trust region, line search and curve search methods.

The validation of the presented algorithm was performed using reference data simulating a complex surface described by an explicit equation. Obtained results show that the proposed method performs well. The algorithm was able to converge even with initial alignment up to 15° in all directions ($x$, $y$ and $z$). In a previous work, the algorithm was also tested on aspherical shapes and the obtained results are promoting.

In the future, the behaviour of the hybrid trust region algorithm will be assessed when used for more complex representations such as NURBS.


## Acknowledgements

The authors sincerely thank the EMPIR organization. The EMPIR initiative is co-funded by the European Union's Horizon 2020 and innovation programme and the EMPIR participating states within EURAMET and the European Union (15SIB01: FreeFORM).


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